or, in component form,
\[
\begin{pmatrix}
  x_{n+1} \\
y_{n+1}
\end{pmatrix} = \begin{pmatrix}
  x_n \\
y_n
\end{pmatrix} + h \begin{pmatrix}
f(t_n, x_n, y_n) \\
g(t_n, x_n, y_n)
\end{pmatrix}.
\]  
(5)

The initial conditions are used to determine \( f_0 \), which is the vector tangent to the graph of the solution \( x = \phi(t) \) in the \( xy \)-plane. We move in the direction of this tangent vector for a time step \( h \) in order to find the next point \( x_1 \). Then we calculate a new tangent vector \( f_1 \), move along it for a time step \( h \) to find \( x_2 \), and so forth.

In a similar way the Runge–Kutta method can be extended to a system. For the step from \( t_n \) to \( t_{n+1} \) we have
\[
x_{n+1} = x_n + \frac{h}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}),
\]
(6)
where
\[
k_{n1} = f(t_n, x_n),
k_{n2} = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}\right),
k_{n3} = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}\right),
k_{n4} = f(t_n + h, x_n + hk_{n3}).
\]
(7)

The formulas for the Adams–Moulton predictor–corrector method as it applies to the initial value problem (1), (2) are given in Problem 9.

The vector equations (3), (4), (6), and (7) are, in fact, valid in any number of dimensions. All that is needed is to interpret the vectors as having \( n \) components rather than two.

**Example 1**

Determine approximate values of the solution \( x = \phi(t) \), \( y = \psi(t) \) of the initial value problem
\[
\begin{align*}
x' &= x - 4y, & y' &= -x + y, \\
x(0) &= 1, & y(0) &= 0,
\end{align*}
\]
(8)
(9)
at the point \( t = 0.2 \). Use the Euler method with \( h = 0.1 \) and the Runge–Kutta method with \( h = 0.2 \). Compare the results with the values of the exact solution:
\[
\begin{align*}
\phi(t) &= \frac{e^{-t} + e^{3t}}{2}, & \psi(t) &= \frac{e^{-t} - e^{3t}}{4}.
\end{align*}
\]
(10)

Let us first use the Euler method. For this problem \( f_n = x_n - 4y_n \) and \( g_n = -x_n + y_n \); hence
\[
\begin{align*}
f_0 &= 1 - (4)(0) = 1, & g_0 &= -1 + 0 = -1.
\end{align*}
\]
Then, from the Euler formulas (4) and (5) we obtain
\[
\begin{align*}
x_1 &= 1 + (0.1)(1) = 1.1, & y_1 &= 0 + (0.1)(-1) = -0.1.
\end{align*}
\]
At the next step
\[
\begin{align*}
f_1 &= 1.1 - (4)(-0.1) = 1.5, & g_1 &= -1.1 + (-0.1) = -1.2.
\end{align*}
\]
Consequently,
\[
\begin{align*}
x_2 &= 1.1 + (0.1)(1.5) = 1.25, & y_2 &= -0.1 + (0.1)(-1.2) = -0.22.
\end{align*}
\]