9.5 Euler's Summation Formula 461

O\((2\pi)^{-2m}\). Of course, we know that this sum is actually a geometric series, equal to \((e^b - e^a)/(e - 1) = (e^b - e^a) \sum_{k \geq 0} B_k/k!\).

If \(f^{(2m)}(x) \geq 0\) for \(a \leq x \leq b\), the integral \(\int_a^b |f^{(2m)}(x)| \, dx\) is just \(f^{(2m-1)}(x)|_{a}^{b}\), so we have

\[
|R_{2m}| \leq \left| \frac{B_{2m}}{(2m)!} f^{(2m-1)}(x)|_{a}^{b} \right|,
\]
in other words, the remainder is bounded by the magnitude of the final term (the term just before the remainder), in this case. We can give an even better estimate if we know that

\[
f^{(2m+2)}(x) \geq 0 \quad \text{and} \quad f^{(2m+4)}(x) \geq 0, \quad \text{for} \quad a \leq x \leq b. \tag{9.79}
\]

For it turns out that this implies the relation

\[
R_{2m} = \theta_m \frac{B_{2m+2}}{(2m+2)!} f^{(2m+1)}(x)|_{a}^{b}, \quad \text{for some} \quad 0 < \theta_m < 1; \tag{9.80}
\]
in other words, the remainder will then lie between 0 and the first discarded term in (9.78) -the term that would follow the final term if we increased \(m\).

Here's the proof: Euler's summation formula is valid for all \(m\), and \(B_{2m+1} = 0\) when \(m > 0\); hence \(R_{2m} = R_{2m+1}\), and the first discarded term must be

\[
R_{2m} = R_{2m+2}.
\]

We therefore want to show that \(R_{2m}\) lies between 0 and \(R_{2m} = R_{2m+2}\); and this is true if and only if \(R_{2m}\) and \(R_{2m+2}\) have opposite signs. We claim that

\[
f^{(2m+2)}(x) \geq 0 \quad \text{for} \quad a \leq x \leq b \implies (-1)^m R_{2m} \geq 0. \tag{9.81}
\]

This, together with (9.79), will prove that \(R_{2m}\) and \(R_{2m+2}\) have opposite signs, so the proof of (9.80) will be complete.

It's not difficult to prove (9.81) if we recall the definition of \(R_{2m+1}\) and the facts we proved about the graph of \(B_{2m+1}(x)\). Namely, we have

\[
R_{2m} = R_{2m+1} = \int_a^b \frac{B_{2m+1}(x)}{(2m+1)!} f^{(2m+1)}(x) \, dx,
\]
and \(f^{(2m+1)}(x)\) is increasing because its derivative \(f^{(2m+2)}(x)\) is positive. (More precisely, \(f^{(2m+1)}(x)\) is nondecreasing because its derivative is nonnegative.) The graph of \(B_{2m+1}(x)\) looks like \((-1)^m\) times a sine wave, so it is geometrically obvious that the second half of each sine wave is more influential than the first half when it is multiplied by an increasing function. This makes \((-1)^m R_{2m+1} \geq 0\), as desired. Exercise 16 proves the result formally.