The values of the exact solution, correct to eight digits, are \( \phi(0.2) = 1.3204248 \) and \( \psi(0.2) = -0.25084701 \). Thus the values calculated from the Euler method are in error by about 0.0704 and 0.0308, respectively, corresponding to percentage errors of about 5.3% and 12.3%.

Now let us use the Runge–Kutta method to approximate \( \phi(0.2) \) and \( \psi(0.2) \). With \( h = 0.2 \) we obtain the following values from Eqs. (7):

\[
\begin{align*}
k_{01} &= \begin{pmatrix} f(1, 0) \\ g(1, 0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \\
k_{02} &= \begin{pmatrix} f(1.1, -0.1) \\ g(1.1, -0.1) \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1.2 \end{pmatrix}; \\
k_{03} &= \begin{pmatrix} f(1.15, -0.12) \\ g(1.15, -0.12) \end{pmatrix} = \begin{pmatrix} 1.63 \\ -1.27 \end{pmatrix}; \\
k_{04} &= \begin{pmatrix} f(1.326, -0.254) \\ g(1.326, -0.254) \end{pmatrix} = \begin{pmatrix} 2.342 \\ -1.580 \end{pmatrix}.
\end{align*}
\]

Then, substituting these values in Eq. (6) we obtain

\[
x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.2 \begin{pmatrix} 9.602 \\ -7.52 \end{pmatrix} = \begin{pmatrix} 1.3200667 \\ -0.25066667 \end{pmatrix}.
\]

These values of \( x_1 \) and \( y_1 \) are in error by about 0.000358 and 0.000180, respectively, with percentage errors much less than one-tenth of 1%.

This example again illustrates the great gains in accuracy that are obtainable by using a more accurate approximation method, such as the Runge–Kutta method. In the calculations we have just outlined, the Runge–Kutta method requires only twice as many function evaluations as the Euler method, but the error in the Runge–Kutta method is about 200 times less than the Euler method.

### PROBLEMS

In each of Problems 1 through 6 determine approximate values of the solution \( x = \phi(t) \), \( y = \psi(t) \) of the given initial value problem at \( t = 0.2, 0.4, 0.6, 0.8, \) and 1.0. Compare the results obtained by different methods and different step sizes.

(a) Use the Euler method with \( h = 0.1 \).
(b) Use the Runge–Kutta method with \( h = 0.2 \).
(c) Use the Runge–Kutta method with \( h = 0.1 \).

1. \( x' = x + y + t, \quad y' = 4x - 2y; \quad x(0) = 1, \quad y(0) = 0 \)
2. \( x' = 2x + ty, \quad y' = xy; \quad x(0) = 1, \quad y(0) = 1 \)
3. \( x' = -tx - y - 1, \quad y' = x; \quad x(0) = 1, \quad y(0) = 1 \)
4. \( x' = x - y + xy, \quad y' = 3x - 2y - xy; \quad x(0) = 0, \quad y(0) = 1 \)
5. \( x' = x(1 - 0.5x - 0.5y), \quad y' = y(-0.25 + 0.5x); \quad x(0) = 4, \quad y(0) = 1 \)
6. \( x' = \exp(-x + y) - \cos x, \quad y' = \sin(x - 3y); \quad x(0) = 1, \quad y(0) = 2 \)
7. Consider the example problem \( x' = x - 4y, \quad y' = -x + y \) with the initial conditions \( x(0) = 1 \) and \( y(0) = 0 \). Use the Runge–Kutta method to solve this problem on the interval \( 0 \leq t \leq 1 \). Start with \( h = 0.2 \) and then repeat the calculation with step sizes \( h = 0.1, 0.05, \ldots \), each half as long as in the preceding case. Continue the process until