Now comes the summing up, as we prepare to conclude this book. We will apply Euler’s summation formula to some interesting and important examples.

**Summation 1: This one is too easy.**

But first we will consider an interesting *unimportant* example, namely a sum that we already know how to do. Let’s see what Euler’s summation formula tells us if we apply it to the telescoping sum

\[ S_n = \sum_{1 \leq k < n} \frac{1}{k(k+1)} = \frac{1}{1} - \frac{1}{n+1} = 1 - \frac{1}{n}. \]

It can’t hurt to embark on our first serious application of Euler’s formula with the asymptotic equivalent of training wheels.

We might as well start by writing the function \( f(x) = \frac{1}{x(x+1)} \) in partial fraction form,

\[ f(x) = \frac{1}{x} - \frac{1}{x+1}, \]

since this makes it easier to integrate and differentiate. Indeed, we have \( f'(x) = -1/x^2 + 1/(x+1)^2 \) and \( f''(x) = 2/x^3 - 2/(x+1)^3 \); in general

\[ f^{(k)}(x) = (-1)^k k! \left( \frac{1}{x^{k+1}} - \frac{1}{(x+1)^{k+1}} \right), \quad \text{for } k \geq 0. \]

Furthermore

\[ \int_1^n f(x) \, dx = \ln x - \ln(x+1) \bigg|_1^n = \ln \frac{2n}{n+1}. \]

Plugging this into the summation formula (9.67) gives

\[ S_n = \ln \frac{2n}{n+1} - \sum_{k=1}^m (-1)^k k! \left( \frac{1}{n^k} - \frac{1}{(n+1)^k} \right) \frac{1}{2^k} + R_m(n), \]

where \( R_m(n) = -\int_1^n B_m(x) \left( \frac{1}{x^{m+1}} - \frac{1}{(x+1)^{m+1}} \right) dx. \)

For example, the right-hand side when \( m = 4 \) is

\[ \ln \frac{2n}{n+1} - \frac{1}{2} \left( \frac{1}{n^4} - \frac{1}{(n+1)^4} \right) - \frac{1}{12} \left( \frac{1}{n^6} - \frac{1}{(n+1)^6} - \frac{3}{4} \right) + \frac{1}{120} \left( \frac{1}{n^8} - \frac{1}{(n+1)^8} \right) + R_4(n). \]