This is kind of a mess; it certainly doesn’t look like the real answer 1 = n⁻¹.

But let’s keep going anyway, to see what we’ve got. We know how to expand
the right-hand terms in negative powers of n up to, say, O(n⁻⁵);

\[
\ln \frac{n}{n+1} = -n^{-1} + \frac{1}{2} n^{-2} - \frac{1}{3} n^{-3} + \frac{1}{4} n^{-4} + O(n^{-5});
\]
\[
\ln \left(1 + \frac{1}{n+1}\right) = n^{-1} - n^{-2} + n^{-3} - n^{-4} + O(n^{-5});
\]
\[
\ln \left(\frac{1}{n+1}\right)^2 = n^{-2} - 2n^{-3} + 3n^{-4} + O(n^{-5});
\]
\[
\frac{1}{(n+1)^3} = n^{-4} + O(n^{-5})
\]

Therefore the terms on the right of our approximation add up to

\[
\ln 2 + \frac{1}{4} + \frac{1}{16} - \frac{1}{128} + (-1) \left(\frac{1}{2} n^{-1} + \left(\frac{1}{3} - \frac{1}{2} \right) n^{-2} + \left(\frac{1}{3} + \frac{1}{2} \right) n^{-3} + \left(\frac{1}{2} - \frac{1}{3} \right) n^{-4} + \frac{1}{120}\right) + R_4(n)
\]
\[
= \ln 2 + \frac{30}{128} - n^{-1} + R_4(n) + O(n^{-5}).
\]

The coefficients of n⁻², n⁻³, and n⁻⁴ cancel nicely, as they should.

If all were well with the world, we would be able to show that R₄(n) is
asymptotically small, maybe O(n⁻⁵), and we would have an approximation
to the sum. But we can’t possibly show this, because we happen to know that
the correct constant term is 1, not ln2 + \frac{19}{128} (which is approximately 0.9978). So R₄(n) is actually equal to \frac{89}{128} = \ln 2 + O(n⁻⁴), but Euler’s summation
formula doesn’t tell us this.

In other words, we lose.

One way to try fixing things is to notice that the constant terms in the
approximation form a pattern, if we let m get larger and larger:

\[
\ln 2 - \frac{1}{2} B_1 - \frac{1}{4} B_2 - \frac{1}{3} B_3 + \frac{1}{4} \cdot \frac{1}{16} B_4 - \frac{1}{3} \cdot \frac{1}{32} B_5 + \cdots
\]

Perhaps we can show that this series approaches 1 as the number of terms
becomes infinite? But no; the Bernoulli numbers get very large. For example,
B₂₂ = \frac{854513}{138} > 6192; therefore |R₂₂(n)| will be much larger than |R₄(n)|.
We lose totally.

There is a way out, however, and this escape route will turn out to be
important in other applications of Euler’s formula. The key is to notice that
R₄(n) approaches a definite limit as n \to \infty:

\[
\lim_{n \to \infty} R_4(n) = - \int_1^\infty B_4(\{x\}) \left(\frac{1}{x^5} - \frac{1}{(x+1)^5}\right) dx = R_4(\infty)
\]