The integral \( \int_{-\infty}^{\infty} B_4(x) f^{(4)}(x) \, dx \) will exist whenever \( f^{(4)}(x) = O(x^2) \) as \( x \to \infty \), and in this case \( f^{(4)}(x) \) surely qualifies. Moreover, we have

\[
R_4(n) = R_4(\infty) + \int_{n}^{\infty} B_4(x) \left( \frac{1}{x^3} - \frac{1}{(x+1)^5} \right) \, dx
= R_4(\infty) + O\left( \int_{n}^{\infty} x^{-8} \, dx \right) = R_4(\infty) + O(n^{-5}).
\]

Thus we have used Euler’s summation formula to prove that

\[
\sum_{1 \leq k < n} \frac{1}{k(k+1)} = \ln 2 + \frac{3}{128} - n^{-1} + R_4(\infty) + O(n^{-5})
= C \cdot n^{-1} + O(n^{-5})
\]

for some constant \( C \). We do not know what the constant is-some other method must be used to establish it -but Euler’s summation formula is able to let us deduce that the constant exists.

Suppose we had chosen a much larger value of \( m \). Then the same reasoning would tell us that

\[
R_4(n) = R_4(m) + O(n^{-m-1}),
\]

and we would have the formula

\[
\sum_{1 \leq k < n} \frac{1}{k(k+1)} = C \cdot n^{-1} + c_2 n^{-2} + c_3 n^{-3} + \ldots + c_m n^{-m} + O(n^{-m-1})
\]

for certain constants \( c_2, c_3, \ldots \). We know that the \( c \)'s happen to be zero in this case; but let’s prove it, just to restore some of our confidence (in Euler’s formula if not in ourselves). The term \( \ln \frac{n}{n+1} \) contributes \((-1)^m/m \) to \( c_m \); the term \((-1)^{m+1} \cdot (B_m/m)n^{-m} \) contributes \((-1)^m \cdot B_m/m \); and the term \((-1)^k (B_k/k)(n+1)^{-k} \) contributes \((-1)^m (m-1)! B_k/k \). Therefore

\[
(-1)^m c_m = \frac{1}{m} - \frac{B_m}{m} + \sum_{k=1}^{m} \left( \frac{m-1}{k-1} \right) \frac{B_k}{k}
= \frac{1}{m} - \frac{B_m}{m} + \frac{1}{m} \sum_{k=1}^{m} \left( \frac{m}{k} \right) B_k = \frac{1}{m} \left( 1 - B_m + B_m(1) - 1 \right).
\]

Sure enough, it’s zero, when \( m > 1 \). We have proved that

\[
\sum_{1 \leq k < n} \frac{1}{k(k+1)} = C \cdot n^{-1} + O(n^{-m-1}), \quad \text{for all } m \geq 1.
\]  

This is not enough to prove that the sum is exactly equal to \( C \cdot n^{-1} \); the actual value may be \( C \cdot n^{-1} + 2^{-n} \) or something. But Euler’s summation