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initial point on the line through \( \xi^{(1)} \), then \( c_2 = 0 \). Consequently, the solution remains on the line through \( \xi^{(1)} \) for all \( t \), and approaches the origin as \( t \to \infty \). Similarly, if the initial point is on the line through \( \xi^{(2)} \), then the solution approaches the origin along that line. In the general situation, it is helpful to rewrite Eq. (5) in the form

\[
x = e^{r_1 t} [c_1 \xi^{(1)} e^{r_1 t} + c_2 \xi^{(2)}].
\]  

(6)

Observe that \( r_1 - r_2 < 0 \). Therefore, as long as \( c_2 \neq 0 \), the term \( c_1 \xi^{(1)} \exp[(r_1 - r_2)t] \) is negligible compared to \( c_2 \xi^{(2)} \) for \( t \) sufficiently large. Thus, as \( t \to \infty \), the trajectory not only approaches the origin, it also tends toward the line through \( \xi^{(2)} \). Hence all solutions approach the critical point tangent to \( \xi^{(2)} \) except for those solutions that start exactly on the line through \( \xi^{(1)} \). Several trajectories are sketched in Figure 9.1.1a. Some typical graphs of \( x_1 \) versus \( t \) are shown in Figure 9.1.1b, illustrating that all solutions exhibit exponential decay in time. The behavior of \( x_2 \) versus \( t \) is similar. This type of critical point is called a **node** or a **nodal sink**.

![FIGURE 9.1.1](image)

**FIGURE 9.1.1** A node; \( r_1 < r_2 < 0 \). (a) The phase plane. (b) \( x_1 \) versus \( t \).

If \( r_1 \) and \( r_2 \) are both positive, and \( 0 < r_2 < r_1 \), then the trajectories have the same pattern as in Figure 9.1.1a, but the direction of motion is away from, rather than toward, the critical point at the origin. In this case \( x_1 \) and \( x_2 \) grow exponentially as functions of \( t \). Again the critical point is called a node, or (often) a **nodal source**.

An example of a node occurs in **Example 2 of Section 7.5** and its trajectories are shown in **Figure 7.5.4**.

**CASE 2** Real Eigenvalues of Opposite Sign. The general solution of Eq. (2) is

\[
x = c_1 \xi^{(1)} e^{r_1 t} + c_2 \xi^{(2)} e^{r_2 t},
\]  

(7)

where \( r_1 > 0 \) and \( r_2 < 0 \). Suppose that the eigenvectors \( \xi^{(1)} \) and \( \xi^{(2)} \) are as shown in **Figure 9.1.2a**. If the solution starts at an initial point on the line through \( \xi^{(1)} \), then it follows that \( c_2 = 0 \). Consequently, the solution remains on the line through \( \xi^{(1)} \) for all \( t \), and since \( r_1 > 0 \), \( \|x\| \to \infty \) as \( t \to \infty \). If the solution starts at an initial point on the line through \( \xi^{(2)} \), then the situation is similar except that \( \|x\| \to 0 \) as \( t \to \infty \) because \( r_2 < 0 \). Solutions starting at other initial points follow trajectories such as those shown in **Figure 9.1.2a**. The positive exponential is the dominant term in Eq. (7) for large \( t \), so eventually all these solutions approach infinity asymptotic to the line determined