by the eigenvector $\xi^{(1)}$ corresponding to the positive eigenvalue $r_1$. The only solutions that approach the critical point at the origin are those that start precisely on the line determined by $\xi^{(2)}$. Figure 9.1.2b shows some typical graphs of $x_1$ versus $t$. For certain initial conditions the positive exponential term is absent from the solution, so $x_1 \to 0$ as $t \to \infty$. For all other initial conditions the positive exponential term eventually dominates and causes $x_1$ to become unbounded. The behavior of $x_2$ is similar. The origin is called a saddle point in this case.

A specific example of a saddle point is in Example 1 of Section 7.5 whose trajectories are shown in Figure 7.5.2.

**CASE 3 Equal Eigenvalues.** We now suppose that $r_1 = r_2 = r$. We consider the case in which the eigenvalues are negative; if they are positive, the trajectories are similar but the direction of motion is reversed. There are two subcases, depending on whether the repeated eigenvalue has two independent eigenvectors or only one.

(a) **Two independent eigenvectors.** The general solution of Eq. (2) is

$$
\mathbf{x} = c_1 \xi^{(1)} e^{rt} + c_2 \xi^{(2)} e^{rt},
$$

where $\xi^{(1)}$ and $\xi^{(2)}$ are the independent eigenvectors. The ratio $x_2/x_1$ is independent of $t$, but depends on the components of $\xi^{(1)}$ and $\xi^{(2)}$, and on the arbitrary constants $c_1$ and $c_2$. Thus every trajectory lies on a straight line through the origin, as shown in Figure 9.1.3a. Typical graphs of $x_1$ or $x_2$ versus $t$ are shown in Figure 9.1.3b. The critical point is called a proper node, or sometimes a star point.

(b) **One independent eigenvector.** As shown in Section 7.8, the general solution of Eq. (2) in this case is

$$
\mathbf{x} = c_1 \xi e^{rt} + c_2 (\xi e^{rt} + \eta e^{rt}),
$$

where $\xi$ is the eigenvector and $\eta$ is the generalized eigenvector associated with the repeated eigenvalue. For large $t$ the dominant term in Eq. (9) is $c_2 \xi e^{rt}$. Thus, as $t \to \infty$, every trajectory approaches the origin tangent to the line through the eigenvector. This is true even if $c_2 = 0$, for then the solution $\mathbf{x} = c_1 \xi e^{rt}$ lies on this line. Similarly, for large negative $t$ the term $c_2 \xi e^{rt}$ is again the dominant one, so as $t \to -\infty$, each trajectory is asymptotic to a line parallel to $\xi$. 

**FIGURE 9.1.2** A saddle point; $r_1 > 0, r_2 < 0.$ (a) The phase plane. (b) $x_1$ versus $t$. 

![Saddle Point Diagram](image-url)