In Chapter 5, equation (5.83), we generalized factorials to arbitrary real α by using a definition
\[
\frac{1}{a!} = \lim_{n \to \infty} \left( \frac{\alpha}{n} \right)^n a^{-n}
\]
suggested by Euler. Suppose a is a large number; then
\[
\ln \alpha! = \lim_{n \to \infty} \left( \alpha \ln n + \ln n! - \sum_{k=1}^{n} \ln (\alpha + k) \right),
\]
and Euler’s summation formula can be used with \( f(x) = \ln(x + a) \) to estimate this sum:
\[
\sum_{k=1}^{n} \ln(k + a) = F_m(\alpha, n) - F_m(\alpha, 0) + R_{2m}(\alpha, n),
\]
\[
F_m(\alpha, x) = (x + \alpha) \ln(x + \alpha) - \frac{\ln(x + \alpha)}{x + \alpha} + \sum_{k=1}^{m} \frac{B_{2k}}{2k(2k - 1)} (x + \alpha)^{2k - 1},
\]
\[
R_{2m}(\alpha, n) = \int_{0}^{n} \frac{B_{2m}(x)}{2m} (x + \alpha)^{2m} dx.
\]
(Here we have used (9.67) with \( a = 0 \) and \( b = n \), then added \( \ln(n + a) = \ln \alpha \) to both sides.) If we subtract this approximation for \( \sum_{k=1}^{n} \ln(k + a) \) from Stirling’s approximation for \( \ln n! \), then add \( \alpha \ln n \) and take the limit as \( n \to \infty \), we get
\[
\ln \alpha! = \alpha \ln \alpha - \alpha + \frac{\ln \alpha}{2} + \sigma
\]
\[
+ \sum_{k=1}^{m} \frac{B_{2k}}{(2k)(2k - 1)\alpha^{2k - 1}} \int_{0}^{\infty} \frac{B_{2m}(x)}{2m} (x + \alpha)^{2m} dx,
\]
because \( \alpha \ln n + n \ln n - n + \frac{1}{2} \ln(n + \alpha) \ln(n + \alpha) + n - \frac{1}{2} \ln(n + \alpha) \to -a \) and the other terms not shown here tend to zero. Thus Stirling’s approximation behaves for generalized factorials (and for the Gamma function \( \Gamma(a + 1) = \alpha! \)) exactly as for ordinary factorials.

**Summation 4: A bell-shaped summand.**

Let’s turn now to a sum that has quite a different flavor:
\[
\Theta_n = \sum_{k} e^{-k}/n
\]
\[
+ e^{-9/n} + e^{-4/n} + e^{1/n} + e^{-1/n} + e^{-4/n} + e^{-9/n} + \ldots
\]