9.1 The Phase Plane: Linear Systems

We introduce the polar coordinates \( r, \theta \) given by
\[
r^2 = x_1^2 + x_2^2, \quad \tan \theta = x_2 / x_1.
\]

By differentiating these equations we obtain
\[
rr' = x_1 x_1' + x_2 x_2', \quad (\sec^2 \theta) \theta' = (x_1 x_2' - x_2 x_1') / x_1^2. \tag{13}
\]

Substituting from Eqs. (12) in the first of Eqs. (13), we find that
\[
r' = \lambda r, \tag{14}
\]
and hence
\[
r = ce^{\lambda t}, \tag{15}
\]
where \( c \) is a constant. Similarly, substituting from Eqs. (12) in the second of Eqs. (13), and using the fact that \( \sec^2 \theta = r^2 / x_1^2 \), we have
\[
\theta' = -\mu. \tag{16}
\]

Hence
\[
\theta = -\mu t + \theta_0, \tag{17}
\]
where \( \theta_0 \) is the value of \( \theta \) when \( t = 0 \).

Equations (15) and (17) are parametric equations in polar coordinates of the trajectories of the system (11). Since \( \mu > 0 \), it follows from Eq. (17) that \( \theta \) decreases as \( t \) increases, so the direction of motion on a trajectory is clockwise. As \( t \to \infty \), we see from Eq. (15) that \( r \to 0 \) if \( \lambda < 0 \) and \( r \to \infty \) if \( \lambda > 0 \). Thus the trajectories are spirals, which approach or recede from the origin depending on the sign of \( \lambda \). Both possibilities are shown in Figure 9.1.5 along with some typical graphs of \( x_1 \) versus \( t \).

The critical point is called a **spiral point** in this case. Frequently, the terms **spiral sink** and **spiral source**, respectively, are used to refer to spiral points whose trajectories approach, or depart from, the critical point.

More generally, it is possible to show that for any system with complex eigenvalues \( \lambda \pm i \mu \), where \( \lambda \neq 0 \), the trajectories are always spirals. They are directed inward or outward, respectively, depending on whether \( \lambda \) is negative or positive. They may be elongated and skewed with respect to the coordinate axes, and the direction of motion may be either clockwise or counterclockwise. While a detailed analysis is moderately difficult, it is easy to obtain a general idea of the orientation of the trajectories directly from the differential equations. Suppose that
\[
\begin{pmatrix}
  \frac{dx}{dt} \\
  \frac{dy}{dt}
\end{pmatrix} =
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}, \tag{18}
\]

has complex eigenvalues \( \lambda \pm i \mu \), and look at the point \((0, 1)\) on the positive \( y \)-axis. At this point it follows from Eqs. (18) that \( \frac{dx}{dt} = b \) and \( \frac{dy}{dt} = d \). Depending on the signs of \( b \) and \( d \), one can infer the direction of motion and the approximate orientation of the trajectories. For instance, if both \( b \) and \( d \) are negative, then the trajectories cross the positive \( y \)-axis so as to move down and into the second quadrant. If \( \lambda < 0 \) also, then the trajectories must be inward-pointing spirals resembling the one in Figure 9.1.6. Another case was given in Example 1 of Section 7.6, whose trajectories are shown in Figure 7.6.2.