For example, we have \( g'(x) = -2xe^{-x^2} \) and \( g''(x) = (4x^2 - 2)e^{-x^2} \); hence

\[
\begin{align*}
f'(x) &= \frac{1}{\sqrt{n}}(-2x)\frac{x}{\sqrt{n}}e^{-x^2/n}, \\
f''(x) &= \frac{1}{n}(4\frac{x}{\sqrt{n}})^2 - 2) e^{-x^2/n}.
\end{align*}
\]

It’s easier to see what’s going on if we work with the simpler function \( g(x) \).

We don’t have to evaluate the derivatives of \( g(x) \) exactly, because we’re only going to be concerned about the limiting values when \( x = \pm \infty \). And for this purpose it suffices to notice that every derivative of \( g(x) \) is \( e^{-x^2} \times \) a polynomial in \( x \):

\[ g^{(k)}(x) = P_k(x)e^{-x^2}, \quad \text{where } P_k \text{ is a polynomial of degree } k. \]

This follows by induction.

The negative exponential \( e^{-x^2} \) goes to zero much faster than \( P_k(x) \) goes to infinity, when \( x \to \pm \infty \), so we have

\[ f^{(k)}(+\infty) = f^{(k)}(-\infty) = 0 \]

for all \( k \geq 0 \). Therefore all of the terms

\[
\sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x) \bigg|_{-\infty}^{+\infty}
\]

vanish, and we are left with the term from \( \int f(x) \, dx \) and the remainder:

\[
\Theta_n = C\sqrt{n} + (-1)^{m+1} \int_{-\infty}^{+\infty} \frac{B_m((x))}{m!} f^{(m)}(x) \, dx
\]

\[
= C\sqrt{n} + \frac{(-1)^{m+1}}{n^{m/2}} \int_{-\infty}^{+\infty} \frac{B_m((x))}{m!} g^{(m)}\left(\frac{x}{\sqrt{n}}\right) \, dx
\]

\[
= C\sqrt{n} + \frac{(-1)^{m+1}}{n^{(m-1)/2}} \int_{-\infty}^{+\infty} \frac{B_m((u\sqrt{n}))}{m!} P_m(u)e^{-u^2} \, du
\]

\[
= C\sqrt{n} + O(n^{1-m}/2)
\]

The 0 estimate here follows since \( |B_m((u\sqrt{n}))| \) is bounded and the integral \( \int_{-\infty}^{+\infty} |P(u)| e^{-u^2} \, du \) exists whenever \( P \) is a polynomial. (The constant implied by this 0 depends on \( m \).)

We have proved that \( \Theta_n = C\sqrt{n} + O(n^{-M}), \) for arbitrarily large \( M \); the difference between \( \Theta_n \) and \( C\sqrt{n} \) is “exponentially small!” Let us therefore determine the constant \( C \) that plays such a big role in the value of \( \Theta_0 \).

One way to determine \( C \) is to look the integral up in a table; but we prefer to know how the value can be derived, so that we can do integrals even