with eigenvalues $\pm i \mu$. Using the same argument as in Case 4 we find that
\[ r' = 0, \quad \theta' = -\mu, \]
and consequently,
\[ r = c, \quad \theta = -\mu t + \theta_0, \]
where $c$ and $\theta_0$ are constants. Thus the trajectories are circles, with center at the origin, that are traversed clockwise if $\mu > 0$ and counterclockwise if $\mu < 0$. A complete circuit about the origin is made in a time interval of length $2\pi/\mu$, so all solutions are periodic with period $2\pi/\mu$. The critical point is called a center.

In general, when the eigenvalues are pure imaginary, it is possible to show (see Problem 19) that the trajectories are ellipses centered at the origin. A typical situation is shown in Figure 9.1.7, which also shows some typical graphs of $x_1$ versus $t$.

By reflecting on these five cases and by examining the corresponding figures, we can make several observations:

1. After a long time each individual trajectory exhibits one of only three types of behavior. As $t \to \infty$, each trajectory either approaches infinity, approaches the critical point $x = 0$, or repeatedly traverses a closed curve, corresponding to a periodic solution, that surrounds the critical point.

2. Viewed as a whole, the pattern of trajectories in each case is relatively simple. To be more specific, through each point $(x_0, y_0)$ in the phase plane there is only one trajectory; thus the trajectories do not cross each other. Do not be misled by the figures, in which it sometimes appears that many trajectories pass through the critical point $x = 0$. In fact, the only solution passing through the origin is the equilibrium solution $x = 0$. The other solutions that appear to pass through the origin actually only approach this point as $t \to \infty$ or $t \to -\infty$.

3. In each case the set of all trajectories is such that one of three situations occurs.
   
   (a) All trajectories approach the critical point $x = 0$ as $t \to \infty$. This is the case if the eigenvalues are real and negative or complex with negative real part. The origin is either a nodal or a spiral sink.
   
   (b) All trajectories remain bounded but do not approach the origin as $t \to \infty$. This is the case if the eigenvalues are pure imaginary. The origin is a center.