The final task seems almost absurdly easy: We will try to find the asymptotic value of

\[ A_n = \sum_k \binom{2n}{k} \]

by using Euler's summation formula.

This is another case where we already know the answer (right?); but it's always interesting to try new methods on old problems, so that we can compare facts and maybe discover something new.

So we think big and realize that the main contribution to \( A_n \) comes from the middle terms, near \( k = n \). It's almost always a good idea to choose notation so that the biggest contribution to a sum occurs near \( k = 0 \), because we can then use the tail-exchange trick to get rid of terms that have large \( |k| \).

Therefore we replace \( k \) by \( n + k \):

\[ A_n = \sum_k \binom{2n}{n+k} = \sum_k \frac{(2n)!}{(n+k)! (n-k)!} \]

Things are looking reasonably good, since we know to approximate \((n \pm k)!\) when \( n \) is large and \( k \) is small.

Now we want to carry out the three-step procedure associated with the tail-exchange trick. Namely, we want to write

\[ \frac{(2n)!}{(n+k)! (n-k)!} = a_k(n) = b_k(n) + O(c_k(n)) \]

for \( k \in D_n \), so that we can obtain the estimate

\[ A_n = \sum_k b_k(n) + O\left( \sum_k a_k(n) \right) + O\left( \sum_k b_k(n) \right) + \sum_k O(c_k(n)) \]

Let us therefore try to estimate \( \binom{2n}{n+k} \) in the region where \( |k| \) is small. We could use Stirling's approximation as it appears in Table 438, but it's easier to work with the logarithmic equivalent in (9.91):

\[
\ln a_k(n) = \ln(2n)! - \ln(n+k)! - \ln(n-k)! = 2n \ln 2n - 2n + \frac{1}{2} \ln 2n + \sigma + O(n^{-1})
\]

\[
- (n+k) \ln(n+k) + n + k - \frac{1}{2} \ln(n+k) - \sigma + \mathcal{O}(n^{-1})
\]

\[
- (n-k) \ln(n-k) + n - k - \frac{1}{2} \ln(n-k) - \sigma + \mathcal{O}(n^{-1})
\]

We want to convert this to a nice, simple \( O \) estimate.

The tail-exchange method allows us to work with estimates that are valid only when \( k \) is in the "dominant" set \( D \). But how should we define \( D \)?