9.1 The Phase Plane: Linear Systems

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\[ 3. \quad \frac{dx}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x \]

\[ 4. \quad \frac{dx}{dt} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x \]

\[ 5. \quad \frac{dx}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x \]

\[ 6. \quad \frac{dx}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x \]

\[ 7. \quad \frac{dx}{dt} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} x \]

\[ 8. \quad \frac{dx}{dt} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} x \]

\[ 9. \quad \frac{dx}{dt} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x \]

\[ 10. \quad \frac{dx}{dt} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} x \]

\[ 11. \quad \frac{dx}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x \]

\[ 12. \quad \frac{dx}{dt} = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} x \]

In each of Problems 13 through 16 determine the critical point \( x = x^0 \), and then classify its type and examine its stability by making the transformation \( x = x^0 + u \).

\[ 13. \quad \frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \]

\[ 14. \quad \frac{dx}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \]

\[ 15. \quad \frac{dx}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 5 \end{pmatrix} \]

\[ 16. \quad \frac{dx}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix} x + \begin{pmatrix} \alpha \\ -\gamma \end{pmatrix}; \quad \alpha, \beta, \gamma, \delta > 0 \]

17. The equation of motion of a spring–mass system with damping (see Section 3.8) is

\[ m \frac{d^2 u}{dt^2} + c \frac{du}{dt} + ku = 0, \]

where \( m, c, \) and \( k \) are positive. Write this second order equation as a system of two first order equations for \( x = u, \ y = du/dt \). Show that \( x = 0, \ y = 0 \) is a critical point, and analyze the nature and stability of the critical point as a function of the parameters \( m, c, \) and \( k \). A similar analysis can be applied to the electric circuit equation (see Section 3.8)

\[ L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0. \]

18. Consider the system \( \mathbf{x}' = A \mathbf{x} \), and suppose that \( A \) has one zero eigenvalue.

(a) Show that \( \mathbf{x} = \mathbf{0} \) is a critical point, and that, in addition, every point on a certain straight line through the origin is also a critical point.

(b) Let \( r_1 = 0 \) and \( r_2 \neq 0 \), and let \( \mathbf{\xi}^{(1)} \) and \( \mathbf{\xi}^{(2)} \) be corresponding eigenvectors. Show that the trajectories are as indicated in Figure 9.1.8. What is the direction of motion on the trajectories?

19. In this problem we indicate how to show that the trajectories are ellipses when the eigenvalues are pure imaginary. Consider the system

\[ \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]

(i)

(a) Show that the eigenvalues of the coefficient matrix are pure imaginary if and only if

\[ a_{11} + a_{22} = 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0. \]

(ii)

(b) The trajectories of the system (i) can be found by converting Eqs. (i) into the single equation

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}. \]

(iii)