This is our approximation, with
\[ b_k(n) = \frac{2^{2n+1/2}}{e^{2/n}} e^{-k^2/n}, \quad c_k(n) = 2^{2n} n^{1+3\epsilon} e^{-k^2/n}. \]

Notice that \( k \) enters \( b_k(n) \) and \( c_k(n) \) in a very simple way. We’re in luck, because we will be summing over \( k \).

The tail-exchange trick tells us that \( \sum_k o_k(n) \) will be approximately \( \sum_k b_k(n) \) if we have done a good job of estimation. Let us therefore evaluate
\[
\sum_k b_k(n) = \frac{2^{2n+1/2}}{e^{2/n}} \sum_k e^{-k^2/n} = \frac{2^{2n+1/2}}{e^{2/n}} \Theta_n = \frac{2^{2n} \sqrt{2n}}{e^{2/n}} \left( 1 + O(n^{-M}) \right)
\]

(Another stroke of luck: We get to use the sum \( \Theta_n \) from the previous example.) This is encouraging, because we know that the original sum is actually
\[
A_1 = \sum_k \left( \frac{2^n}{k} \right) = (1 + 1)^{2n} = 2^{2n}.
\]

Therefore it looks as if we will have \( e^\sigma = \sqrt{2n} \), as advertised.

But there’s a catch: We still need to prove that our estimates are good enough. So let’s look first at the error contributed by \( c_k(n) \):
\[
\Sigma_c(n) = \sum_{|k| \leq n^{1/2+\epsilon}} 2^{2n} n^{1+3\epsilon} e^{-k^2/n} \leq 2^{2n} n^{-1+3\epsilon} \Theta_n = O(2^{2n} n^{-1/2+3\epsilon}),
\]

Good; this is asymptotically smaller than the previous sum, if \( 3\epsilon < \frac{1}{2} \).

Next we must check the tails. We have
\[
\sum_{k > n^{1/2+\epsilon}} e^{-k^2/n} < \exp(-|n^{1/2+\epsilon}|^2/n) \{ 1 + e^{-1/n} + e^{-2/n} + \ldots \} = O(e^{-n^{2\epsilon}}) \cdot O(n),
\]

which is \( O(n^{-M}) \) for all \( M \); so \( \sum_{k \in D_0} b_k(n) \) is asymptotically negligible. (We chose the cutoff at \( n^{1/2+\epsilon} \) just so that \( e^{-k^2/n} \) would be exponentially small outside of \( D_0 \). Other choices like \( n^{1/2} \log n \) would have been good enough too, and the resulting estimates would have been slightly sharper, but the formulas would have come out more complicated. We need not make the strongest possible estimates, since our main goal is to establish the value of the constant \( o \).) Similarly, the other tail
\[
\sum_{k > n^{1/2+\epsilon}} \left( \frac{2^n}{n+k} \right)
\]