is bounded by $2n$ times its largest term, which occurs at the cutoff point $k \approx n^{1/2+\epsilon}$. This term is known to be approximately $b_k(n)$, which is exponentially small compared with $A$; and an exponentially small multiplier wipes out the factor of $2n$.

Thus we have successfully applied the tail-exchange trick to prove the estimate

$$2^{2n} = \sum_k \binom{2n}{k} e^{2\sqrt{2\pi \phi n} + O(2^{2n} n^{-1/2+\epsilon})}, \quad \text{if } 0 < \epsilon < \frac{1}{6}. \quad (9.99)$$

We may choose $\epsilon = \frac{1}{8}$ and conclude that

$$\sigma = \frac{1}{2} \ln 2\pi.$$

QED.

**Exercises**

**Warmups**

1. Prove or disprove: If $f_1(n) \prec g_1(n)$ and $f_2(n) \prec g_2(n)$, then we have $f_1(n) + f_2(n) \prec g_1(n) + g_2(n)$.

2. Which function grows faster:
   a. $n^{\ln n}$ or $(\ln n)^n$?
   b. $n^{\ln n \ln n}$ or $(\ln n)!$?
   c. $(n!)!$ or $(\lfloor n - 1 \rfloor)! (\lfloor n - 1 \rfloor)!$?
   d. $F_{2n}$ or $F_{n}$?

3. What’s wrong with the following argument? “Since $n = O(n)$ and $2n = O(n)$ and so on, we have $\sum_{k=1}^{n} kn = \sum_{k=1}^{n} O(n) = O(n^2)$.”

4. Give an example of a valid equation that has $O$-notation on the left but not on the right. (Do not use the trick of multiplying by zero; that’s too easy.) Hint: Consider taking limits.

5. Prove or disprove: $O(f(n) + g(n)) = f(n) + O(g(n))$ if $f(n)$ and $g(n)$ are positive for all $n$. (Compare with $(g.27)$.)

6. Multiply $(\ln n + y + O(1/n))$ by $(n + O(\sqrt{n}))$, and express your answer in $O$-notation.

7. Estimate $\sum_{k \geq 0} e^{-k/n}$ with absolute error $O(n^{-1})$.

**Basics**

8. Give an example of functions $f(n)$ and $g(n)$ such that none of the three relations $f(n) \prec g(n)$, $f(n) \succ g(n)$, $f(n) \asymp g(n)$ is valid, although $f(n)$ and $g(n)$ both increase monotonically to $\infty$. 

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Thanks for reading this, hope it comes in handy.

-The authors