Prove \((9.22)\) rigorously by showing that the left side is a subset of the right side, according to the set-of-functions definition of \(O\).

Prove or disprove: \(\cos O(x) = 1 + O(x^2)\) for all real \(x\).

Prove or disprove: \(O(x+y)^2 = O(x^2) + O(y^2)\).

Prove that
\[
1 + \frac{2}{n} + O(n^{-2}) = \left(1 + \frac{2}{n}\right)(1 + O(n^{-2})),
\]
as \(n \to \infty\).

Evaluate \((n + 2 + O(n^{-1}))^n\) with relative error \(O(n^{-1})\).

Prove that \((n + \alpha)^{n+\beta} = n^{n+\beta} e^{\alpha(1+\alpha(\beta - \frac{1}{2}\alpha)n^{-1} + O(n^{-2}))}\).

Give an asymptotic formula for the “middle” trinomial coefficient \(\binom{n}{n/2, n/2}\), correct to relative error \(O(n^{-3})\).

Show that if \(B(1 - x) = -B(x)\), we have
\[
\int_{0}^{b} B([x]) f(x) \, dx = 0
\]
if we assume also that \(f''(x) \geq 0\) for \(a \leq x \leq b\).

Use generating functions to show that \(B_m(\frac{1}{2}) = (2^{1-m} - 1)B_m\), for all \(m \geq 0\).

Find \(\sum_k \binom{n}{k}^a\) with relative error \(O(n^{-1/4})\), when \(a > 0\).

**Homework exercises**

Use a computer to compare the left and right sides of the approximations in Table 438, when \(n = 10\), \(z = \alpha = 0.1\), and \(O(f(n)) = O(f(z)) = 0\).

Prove or disprove the following estimates, as \(n \to \infty\):

a \(O\left(\left(\frac{n^2}{\log n}\right)^{1/2}\right) = O(\sqrt{n})\).

b \(e^{(1+O(1/n))^2} = e + O(1/n)\).

c \(n! = O((1-1/n)^n n^n)\).

Equation \((9.48)\) gives the nth prime with relative error \(O(\log n)^{-2}\). Improve the relative error to \(O(\log n)^{-3}\) by starting with another term of \((9.31)\) in \((9.46)\).

Improve \((9.54)\) to \(O(n^{-3})\).

Push the approximation \((9.62)\) further, getting absolute error \(O(n^{-3})\). Hint: Let \(g_n = c/(n + 1) (n + 2) + h_n\); what recurrence does \(h_n\) satisfy?