39 Evaluate \( \sum_{0 \leq k < n} \ln(n - k)/(\ln n)^k/k! \) with absolute error \( O(n^{-1}) \). Hint: Show that the terms for \( k \geq 10 \ln n \) are negligible.

40 Let \( m \) be a (fixed) positive integer. Evaluate \( \sum_{k=1}^{n} (-1)^k H_k^m \) with absolute error \( O(1) \).

41 Evaluate the “Fibonacci factorial” \( \prod_{k=1}^{n} F_k \) with relative error \( O(n^{-1}) \) or better. Your answer may involve a constant whose value you do not know in closed form.

42 Let \( \alpha \) be a constant in the range \( 0 < \alpha < \frac{1}{2} \). We’ve seen in previous chapters that there is no general closed form for the sum \( \sum_{k \leq \alpha n} \binom{n}{k} \).

Show that there is, however, an asymptotic formula

\[
\sum_{k \leq \alpha n} \binom{n}{k} = 2^{nH(\alpha) - \frac{1}{2} \log n + O(1)} ,
\]

where \( H(\alpha) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} \). Hint: Show that \( \binom{n}{k} \leq \frac{\alpha}{1 - \alpha} \binom{n}{k-1} \) for \( 0 < k \leq \alpha n \).

43 Let \( C_n \), the number of ways to change \( n \) cents (as considered in Chapter 7) is asymptotically \( cn^4 + O(n^3) \) for some constant \( c \). What is that constant?

44 Prove that

\[
\chi^{1/2} = \chi^{1/2} \left[ \frac{1/2}{1/2} \right] - \chi^{-1/2} \left[ \frac{1/2}{-1/2} \right] + \chi^{-3/2} \left[ \frac{1/2}{-3/2} \right] + O(\chi^{-5/2})
\]

as \( \chi \to \infty \). (Recall the definition \( \chi^{1/2} = \chi! / [\chi^{1/2}]! \) in (5.88), and the definition of generalized Stirling numbers in Table 258.)

45 Let \( \alpha \) be an irrational number between 0 and 1. Chapter 3 discusses the quantity \( D(\alpha, n) \), which measures the maximum discrepancy by which the fractional parts \( \{k\alpha\} \) for \( 0 \leq k < n \) deviate from a uniform distribution. The recurrence

\[
D(\alpha, n) \leq D(\lfloor \alpha^{-1} \rfloor, \lfloor \alpha n \rfloor) + \alpha^{-1} + 2
\]

was proved in (3.31); we also have the obvious bounds

\[
0 \leq D(\alpha, n) \leq n.
\]

Prove that \( \lim_{n \to \infty} D(\alpha, n)/n = 0 \). Hint: Chapter 6 discusses continued fractions.