46 Show that the Bell number \( b_n = e^{-1} \sum_{k \geq 0} \frac{k^n}{k!} \) of exercise 7.15 is asymptotically equal to

\[ m(n)^n e^{m(n) - n - 1/2} / \sqrt{\ln n}, \]

where \( m(n) \) is \( m(n) = n - \frac{1}{2} \), and estimate the relative error in this approximation.

47 Let \( m \) be an integer \( \geq 2 \). Analyze the two sums

\[ \sum_{k=1}^{n} \left\lfloor \log_m n \right\rfloor \quad \text{and} \quad \sum_{k=1}^{n} \left\lfloor \log_m n \right\rfloor ; \]

which is asymptotically closer to \( \log_m n! \) ?

48 Consider a table of the harmonic numbers \( H_k \) for \( 1 \leq k \leq n \) in decimal notation. The \( k \)th entry \( \hat{H}_k \) has been correctly rounded to \( d_k \) significant digits, where \( d_k \) is just large enough to distinguish this value from the values of \( H_{k-1} \) and \( H_{k+1} \). For example, here is an extract from the table, showing five entries where \( H_k \) passes 10:

\[
\begin{array}{|c|c|c|c|}
\hline
k & H_k & \hat{H}_k & d_k \\
\hline
12364 & 9.99980041- & 9.9998 & 5 \\
12365 & 9.99988128+ & 9.9999 & 5 \\
12366 & 9.99996215- & 9.99996 & 6 \\
12367 & 10.00004301- & 10.0000 & 6 \\
12368 & 10.00012386+ & 10.0001 & 6 \\
\hline
\end{array}
\]

Estimate the total number of digits in the table, \( \sum_{k=1}^{n} d_k \), with an absolute error of 0 (n).

49 In Chapter 6 we considered the tale of a worm that reaches the end of a stretching band after \( n \) seconds, where \( H_{n-1} < 100 \leq H_n \). Prove that if \( n \) is a positive integer such that \( H_{n-1} \leq 100 \leq H_n \), then

\[ \lfloor e^{n-\gamma} \rfloor \leq n \leq \lceil e^{n-\gamma} \rceil. \]

50 Venture capitalists in Silicon Valley are being offered a deal giving them a chance for an exponential payoff on their investments: For an \( n \) million dollar investment, where \( n \geq 2 \), the GKP consortium promises to pay up to \( N \) million dollars after one year, where \( N = 10^n \). Of course there's some risk; the actual deal is that GKP pays \( k \) million dollars with probability \( 1/(k^2 H_n^{(2)}) \), for each integer \( k \) in the range \( 1 \leq k \leq N \). (All payments are in megabucks, that is, in exact multiples of $1,000,000; the payoff is determined by a truly random process.) Notice that an investor always gets at least a million dollars back.