480 ASYMPTOTICS

a What is the asymptotic expected return after one year, if \( n \) million dollars are invested? (In other words, what is the mean value of the payment?) Your answer should be correct within an absolute error of \( O(10^{-n}) \) dollars.

b What is the asymptotic probability that you make a profit, if you invest \( n \) million? (In other words, what is the chance that you get back more than you put in?) Your answer here should be correct within an absolute error of \( O(n^{-3}) \).

Bonus problems

51 Prove or disprove: \( \int_n^\infty O(x^{-2}) \, dx = O(n^{-1}) \) as \( n \to \infty \).

52 Show that there exists a power series \( A(z) = \sum_{k \geq 0} a_k z^k \), convergent for all complex \( z \), such that

\[
A(n) \approx n^n \int_0^\infty \frac{1}{x^n} \, dx.
\]

53 Prove that if \( f(x) \) is a function whose derivatives satisfy

\[
f'(x) \geq 0, \quad -f''(x) \leq 0, \quad f'''(x) \leq 0, \quad \ldots \quad (-1)^m f^{(m+1)}(x) \leq 0
\]

for all \( x \geq 0 \), then we have

\[
f(x) = f(0) + \frac{f'(0)}{1!} x + \cdots + \frac{f^{(m-1)}(0)}{(m-1)!} x^{m-1} + O(x^m), \quad \text{for } x \geq 0.
\]

In particular, the case \( f(x) = -\ln(1+x) \) proves (9.64) for all \( k, n > 0 \).

54 Let \( f(x) \) be a positive, differentiable function such that \( xf'(x) \leq f(x) \) as \( x \to \infty \). Prove that

\[
\sum_{k \geq n} \frac{f(k)}{k^{1+\alpha}} = O\left( \frac{f(n)}{n^\alpha} \right), \quad \text{if } \alpha > 0.
\]

Hint: Consider the quantity \( f(k - 1/2)/(k - 1/2)^\alpha \).

55 Improve (9.99) to relative error \( O(n^{-3/2+\epsilon}) \).

56 The quantity \( Q(n) = 1 + \frac{n-1}{n} + \frac{n-2}{n^2} + \cdots + \sum_{k \geq 1} \frac{n^k}{n^k} \) occurs in the analysis of many algorithms. Find its asymptotic value, with absolute error \( o(1) \).

57 An asymptotic formula for Golomb’s sum \( \sum_{k \geq 1} 1/k[1 + \log_k k]^2 \) is derived in (9.54). Find an asymptotic formula for the analogous sum without floor brackets, \( \sum_{k \geq 1} 1/k(1 + \log_k k)^2 \). Hint: We have \( \int_0^\infty x e^{-ux} k^{-tu} \, du = 1/(1 + t \ln k)^2 \).