Hint: Since the trajectory is closed, there exists at least one point \((x_0, y_0)\) such that \(\phi(t_0) = x_0, \psi(t_0) = y_0\) and a number \(T > 0\) such that \(\phi(t_0 + T) = x_0, \psi(t_0 + T) = y_0\). Show that \(x = \Phi(t) = \phi(t + T)\) and \(y = \Psi(t) = \psi(t + T)\) is a solution and then use the existence and uniqueness theorem to show that \(\Phi(t) = \phi(t)\) and \(\Psi(t) = \psi(t)\) for all \(t\).

9.3 Almost Linear Systems

In Section 9.1 we gave an informal description of the stability properties of the equilibrium solution \(x = 0\) of the two-dimensional linear system

\[
x' = Ax.
\]

The results are summarized in Table 9.1.1. Recall that we required that \(\det A \neq 0\), so that \(x = 0\) is the only critical point of the system (1). Now that we have defined the concepts of asymptotic stability, stability, and instability more precisely, we can restate these results in the following theorem.

**Theorem 9.3.1**

The critical point \(x = 0\) of the linear system (1) is asymptotically stable if the eigenvalues \(r_1, r_2\) are real and negative or have negative real part; stable, but not asymptotically stable, if \(r_1\) and \(r_2\) are pure imaginary; unstable if \(r_1\) and \(r_2\) are real and either is positive, or if they have positive real part.

It is apparent from this theorem or from Table 9.1.1 that the eigenvalues \(r_1, r_2\) of the coefficient matrix \(A\) determine the type of critical point at \(x = 0\) and its stability characteristics. In turn, the values of \(r_1\) and \(r_2\) depend on the coefficients in the system (1). When such a system arises in some applied field, the coefficients usually result from the measurements of certain physical quantities. Such measurements are often subject to small uncertainties, so it is of interest to investigate whether small changes (perturbations) in the coefficients can affect the stability or instability of a critical point and/or significantly alter the pattern of trajectories.

Recall that the eigenvalues \(r_1, r_2\) are the roots of the polynomial equation

\[
\det(A - rI) = 0.
\]

It is possible to show that small perturbations in some or all the coefficients are reflected in small perturbations in the eigenvalues. The most sensitive situation occurs when \(r_1 = i\mu\) and \(r_2 = -i\mu\), that is, when the critical point is a center and the trajectories are closed curves surrounding it. If a slight change is made in the coefficients, then the eigenvalues \(r_1\) and \(r_2\) will take on new values \(r_1' = \lambda' + i\mu'\) and \(r_2' = \lambda' - i\mu'\), where \(\lambda'\) is small in magnitude and \(\mu' \approx \mu\) (see Figure 9.3.1). If \(\lambda' \neq 0\), which almost always occurs, then the trajectories of the perturbed system are spirals, rather than closed curves. The system is asymptotically stable if \(\lambda' < 0\), but unstable if \(\lambda' > 0\). Thus, in the case of a center, small perturbations in the coefficients may well change a stable system into an unstable one, and in any case may be expected to alter radically the pattern of trajectories in the phase plane (see Problem 25).