Bertrand’s postulate there is a prime \( p \) between \( n/2 \) and \( n \). We can also assume that \( j > n/2 \), since \( q^j = L(n) + 1 \) leaves \( j = n + 1 \) if and only if \( q \) leaves \( j \). Choose \( q \) so that \( q \equiv 1 \pmod{L(n)/p} \) and \( q \equiv j + 1 \pmod{n} \). The people are now removed in order \( 1, 2, \ldots, n, j + 1, j + 2, \ldots, n, n-p+1, \ldots, j-1 \).

1.24 The only known examples are: \( X_n = a/X_{n-1} \), which has period 2; R. C. Lyness’s recurrence of period 5 in exercise 8; H. Todd’s recurrence \( X_n = (1 + X_n + X_{n-2})/X_{n-3} \), which has period 8; and recurrences derived from these by substitutions of the form \( Y_n = \alpha X_{n-1} \). An exhaustive search by Bill Gosper turned up no nontrivial solutions of period 4 when \( k = 2 \). A partial theory has been developed by Lyness [210] and by Kurshan and Gopinath [189]. An interesting example of another type, with period 9 when the starting values are real, is the recurrence \( X_n = |X_{n-1} - X_{n-2} | \) discovered by Morton Brown [38]. Nonlinear recurrences having any desired period \( \geq 5 \) can be based on continuants [55].

1.25 If \( T_k^{[1]}(n) \) denotes the minimum number of moves needed to transfer \( n \) disks with \( k \) auxiliary pegs (hence \( T_k^{[1]}(n) = T_n \) and \( T_k^{[2]}(n) = W_n \)), we have \( T_k^{[1]}(n) \leq 2T_k^{[1]}(\binom{n-1}{k}) + T^{[1]}(\binom{n}{k+1}) \). No examples \( (n, k) \) are known where this inequality fails to be an equality. When \( k \) is small compared with \( n \), the formula \( 2^{n-k} \binom{n-1}{k-1} \) gives a convenient (but non-optimum) upper bound on \( T_k^{[1]}(n) \).

1.26 The execution-order permutation can be computed in \( O(n \log n) \) steps for all \( m \) and \( n \) [175, exercises 5.1.1-2 and 5.1.1-5]. Bjorn Poonen [241] has proved that non-Josephus sets with exactly four “bad guys” exist whenever \( n \equiv 0 \pmod{3} \) and \( n \geq 9 \); in fact, the number of such sets is at least \( e \binom{n}{4} \) for some \( e > 0 \). He also found by extensive computations that the only other \( n < 24 \) with non-Josephus sets is \( n = 20 \), which has 236 such sets with \( k = 14 \) and two with \( k = 13 \). (One of the latter is \( \{1, 2, 3, 4, 5, 6, 7, 8, 11, 14, 15, 16, 17\} \); the other is its reflection with respect to 21.) There is a unique non-Josephus set with \( n = 15 \) and \( k = 9 \), namely \( \{3, 4, 5, 6, 8, 10, 11, 12, 13\} \).

2.1 There’s no agreement about this; three answers are defensible: (1) We can say that \( \sum_{k=m}^n q_k \) is always equivalent to \( \sum_{m \leq k \leq n} q_k \); then the stated sum is zero. (2) A person might say that the given sum is \( q_4 + q_3 + q_2 + q_1 + q_0 \), by summing over decreasing values of \( k \). But this conflicts with the generally accepted convention that \( \sum_{k=1}^n q_k = 0 \) when \( n = 0 \). (3) We can say that \( \sum_{k=m}^n q_k = \sum_{k \leq m} q_k - \sum_{k \leq m} q_k \); then the stated sum is \( -q_1 - q_2 - q_3 \). This convention may appear strange, but it obeys the useful law \( \sum_{k=a}^b + \sum_{k=b+1}^c = \sum_{k=a}^c \) for all \( a, b, c \).

It’s best to use the notation \( \sum_{k=m}^n q_k \) only when \( n - m \geq -1 \); then both conventions (1) and (3) agree.