“It is a profoundly erroneous truism, repeated by all copybooks and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.”

-A. N. Whitehead

2.15 The first step replaces \( k(k + 1) \) by \( 2 \sum_{1 \leq j \leq k} j \). The second step gives
\[
\Theta_n + \Box_n = (\sum_{k=1}^n k)^2 + \Theta_n.
\]

2.16 \( x^{\alpha} (x - m)^{\alpha + n} = x^{\alpha + n} (x - m)^\alpha \), by (2.52).

2.17 Use induction, for the first two \( = \theta_\alpha \) and (2.52) for the third. The second line follows from the first.

2.18 Use the facts that \((\Re z)^+ \leq |z|, (\Re z)^- \leq |z|, (\Im z)^+ \leq |z|, (\Im z)^- \leq |z|\), and \( |z| \leq (\Re z)^+ + (\Re z)^- + (\Im z)^+ + (\Im z)^- \).

2.19 Multiply both sides by \( 2^{n-1}/n! \) and let \( S_n = 2^n T_n / n! = S_{n-1} + 3 \cdot 2^{n-1} = 3(2^n - 1) + S_0 \). The solution is \( T_n = 3 \cdot n! + n!/2^n - 1 \). (We’ll see in Chapter 4 that \( T_n \) is an integer only when \( n \) is 0 or a power of 2.)

2.20 The perturbation method gives
\[
S_n + (n + 1)H_{n+1} = S_n + \left( \sum_{0 \leq k \leq n} H_k \right) + n + 1.
\]

2.21 Extracting the final term of \( S_{n+1} \) gives \( S_{n+1} = 1 = S_n \); extracting the first term gives
\[
S_{n+1} = (-1)^{n+1} + \sum_{1 \leq k \leq n+1} (-1)^{n+1-k} = (-1)^{n+1} + \sum_{0 \leq k \leq n} (-1)^{n-k}
= (-1)^{n+1} + S_n.
\]
Hence \( 2S_n = 1 + (-1)^n \) and we have \( S_n = [n \text{ is even}] \). Similarly, we find
\[
T_{n+1} = n + 1 - T_n = \sum_{k=0}^n (-1)^{n-k}(k + 1) = T_n + S_n,
\]
hence \( 2T_n = n + 1 - S_n \) and we have \( T_n = \frac{1}{2} (n + [n \text{ is odd}]) \). Finally, the same approach yields
\[
U_{n+1} = (n + 1)^2 - U_n = U_n + 2T_n + S_n
= U_n + n + [n \text{ is odd}] + [n \text{ is even}]
= U_n + n + 1.
\]
Hence \( U_n \) is the triangular number \( \frac{1}{2} (n + 1)n \).

2.22 Twice the sum gives a “vanilla” sum over \( 1 \leq j, k \leq n \), which splits into three sums that can be handled easily.

2.23 (a) This approach gives four sums that evaluate to \( 2n + H_n = 2n + \left( H_n + \frac{1}{n+1} \right) \). (It would have been easier to replace the summand by \( 1/(k+1)/(k+1) \).) (b) Let \( u(x) = 2x + 1 \) and \( Au(x) = 1/x(x + 1) = (x - 1)^{-2} \); then \( Au(x) = 2 \) and \( v(x) = -(x - 1)^{-2} = -1/x \). The answer is \( 2H_n = \frac{n}{n(n+1)} \).