Let us now consider the case \( \gamma^2 - 4\omega^2 < 0 \), corresponding to small damping, in more detail. The direction of motion on the spirals near \((0, 0)\) can be obtained directly from Eqs. (8). Consider the point at which a spiral intersects the positive \(y\)-axis \((x = 0\) and \(y > 0\)). At such a point it follows from Eqs. (8) that \(dx/dt > 0\). Thus the point \((x, y)\) on the trajectory is moving to the right, so the direction of motion on the spirals is clockwise.

The behavior of the pendulum near the critical points \((\pm n\pi, 0)\), with \(n\) even, is the same as its behavior near the origin. We expect this on physical grounds since all these critical points correspond to the downward equilibrium position of the pendulum. The conclusion can be confirmed by repeating the analysis carried out above for the origin. Figure 9.3.3 shows the clockwise spirals at a few of these critical points.

Now let us consider the critical point \((\pi, 0)\). Here the nonlinear equations (8) are approximated by the linear system (17), whose eigenvalues are

\[
\begin{align*}
r_1, r_2 &= -\gamma \pm \sqrt{\gamma^2 + 4\omega^2} \\
&= -\gamma \pm \sqrt{\gamma^2 + 4\omega^2}.
\end{align*}
\]

One eigenvalue \((r_1)\) is positive and the other \((r_2)\) is negative. Therefore, regardless of the amount of damping, the critical point \(x = \pi, y = 0\) is an unstable saddle point of both the linear system (17) and of the almost linear system (8).

To examine the behavior of trajectories near the saddle point \((\pi, 0)\) in more detail we write down the general solution of Eqs. (17), namely,

\[
\begin{pmatrix} u \\ v \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ r_1 \end{pmatrix} e^{r_1 t} + C_2 \begin{pmatrix} 1 \\ r_2 \end{pmatrix} e^{r_2 t},
\]

where \(C_1\) and \(C_2\) are arbitrary constants. Since \(r_1 > 0\) and \(r_2 < 0\), it follows that the solution that approaches zero as \(t \to \infty\) corresponds to \(C_1 = 0\). For this solution \(v/u = r_2\), so the slope of the entering trajectories is negative; one lies in the second quadrant \((C_2 < 0)\) and the other lies in the fourth quadrant \((C_2 > 0)\). For \(C_2 = 0\) we obtain the pair of trajectories "exiting" from the saddle point. These trajectories have slope \(r_1 > 0\); one lies in the first quadrant \((C_1 > 0)\) and the other lies in the third quadrant \((C_1 < 0)\).

The situation is the same at other critical points \((n\pi, 0)\) with \(n\) odd. These all correspond to the upward equilibrium position of the pendulum, so we expect them to be unstable. The analysis at \((\pi, 0)\) can be repeated to show that they are saddle points oriented in the same way as the one at \((\pi, 0)\). Diagrams of the trajectories in the neighborhood of two saddle points are shown in Figure 9.3.4.

![Figure 9.3.3](image_url)  
**Figure 9.3.3** Asymptotically stable spiral points for the damped pendulum.