The trajectories that enter the saddle points separate the phase plane into regions. Such a trajectory is called a separatrix. Each region contains exactly one of the asymptotically stable spiral points. The initial conditions on $\theta$ and $d\theta/dt$ determine the position of an initial point $(x, y)$ in the phase plane. The subsequent motion of the pendulum is represented by the trajectory passing through the initial point as it spirals toward the asymptotically stable critical point in that region. The set of all initial points from which trajectories approach a given asymptotically stable critical point is called the basin of attraction or the region of asymptotic stability for that critical point. Each asymptotically stable critical point has its own basin of attraction, which is bounded by the separatrices through the neighboring unstable saddle points. The basin of attraction for the origin is shown in blue in Figure 9.3.5. Note that it is mathematically possible (but physically unrealizable) to choose initial conditions on a separatrix so that the resulting motion leads to a balanced pendulum in a vertically upward position of unstable equilibrium.

An important difference between nonlinear autonomous systems and the linear systems discussed in Section 9.1 is illustrated by the pendulum equations. Recall that the linear system (1) has only the single critical point $x = 0$ if $\det A \neq 0$. Thus, if the origin is asymptotically stable, then not only do trajectories that start close to the origin approach it, but, in fact, every trajectory approaches the origin. In this case the critical point $x = 0$ is said to be globally asymptotically stable. This property of linear systems is not, in general, true for nonlinear systems. For nonlinear systems an important question is to determine (or to estimate) the basin of attraction for each asymptotically stable critical point.

PROBLEMS

In each of Problems 1 through 4, verify that $(0, 0)$ is a critical point, show that the system is almost linear, and discuss the type and stability of the critical point $(0, 0)$ by examining the corresponding linear system.

1. $dx/dt = x - y^2$, $dy/dt = x - 2y + x^2$
2. $dx/dt = -x + y + 2xy$, $dy/dt = -4x - y + x^2 - y^2$
3. $dx/dt = (1 + x)\sin y$, $dy/dt = 1 - x - \cos y$
4. $dx/dt = x + y^2$, $dy/dt = x + y$

In each of Problems 5 through 16:
(a) Determine all critical points of the given system of equations.
(b) Find the corresponding linear system near each critical point.
(c) Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?
(d) Draw a phase portrait of the nonlinear system to confirm your conclusions, or to extend them in those cases where the linear system does not provide definite information about the nonlinear system.

5. $dx/dt = (2 + x)(y - x)$, $dy/dt = (4 - x)(y + x)$
6. $dx/dt = x - x^2 - xy$, $dy/dt = 3y - xy - 2y^2$
7. $dx/dt = 1 - y$, $dy/dt = x^2 - y^2$
8. $dx/dt = x - x^2 - xy$, $dy/dt = \frac{1}{4}y - \frac{1}{4}y^2 - \frac{1}{2}xy$
9. $dx/dt = -(x - y)(1 - x - y)$, $dy/dt = x(2 + y)$
10. $dx/dt = x + x^2 + y^2$, $dy/dt = y - xy$
11. $dx/dt = 2x + y + xy^3$, $dy/dt = x - 2y - xy$