A ANSWERS TO EXERCISES

There is one solution for each way to write $2100 = x \cdot y$ where $x$ is even and $y$ is odd; we let $a = \frac{1}{2} |x - y| + \frac{1}{2}$ and $b = \frac{1}{2} (x + y) + \frac{1}{2}$. So the number of solutions is the number of divisors of $3 \cdot 5^2 \cdot 7$, namely 12. In general, there are $\prod_{p>2} (n_p + 1)$ ways to represent $\prod_p p^{n_p}$, where the products range over primes.

2.31 $\sum_{j \geq 1} j^k = \sum_{j \geq 1} 1/j^2 (1 - 1/j) = \sum_{j \geq 1} 1/j (1 - j^{-1})$. These second sums, similarly, $3/4$.

2.32 If $2n \leq x < 2n + 1$, the sums are $0 + \ldots + n + (x - n - 1) + \ldots + (x - 2n) = n(x-n) = (x-1) + (x-3) + \ldots + (x-2n+1)$. If $2n - 1 \leq x < 2n$, they are, similarly, both equal to $\operatorname{nl}(x-n)$. (Looking ahead to Chapter 3, the formula $\left\lfloor \frac{1}{2} (x + 1) \right\rfloor (x - \left\lfloor \frac{1}{2} (x + 1) \right\rfloor)$ covers both cases.)

2.33 If $K$ is empty, $\bigwedge_{k \in K} \alpha_k = \infty$. The basic analogies are:

$$\sum_{k \in K} c \alpha_k = c \sum_{k \in K} \alpha_k \quad \longleftrightarrow \quad \bigwedge_{k \in K} \lfloor c + \alpha_k \rfloor = c + \bigwedge_{k \in K} \alpha_k$$

$$\sum_{k \in K} (\alpha_k + b_k) = \sum_{k \in K} \alpha_k + \sum_{k \in K} b_k \quad \longleftrightarrow \quad \bigwedge_{k \in K} \min(\alpha_k, b_k) = \min(\bigwedge_{k \in K} \alpha_k, \bigwedge_{k \in K} b_k)$$

$$\sum_{k \in K} \alpha_k = \sum_{p | k \in K} \alpha_p | k| \quad \longleftrightarrow \quad \bigwedge_{k \in K} \alpha_k = \bigwedge_{p | k \in K} \alpha_p | k|$$

$$\sum_{k \in K} \alpha_{i,k} = \sum_{i \in I, k \in K} \alpha_{i,k} \quad \longleftrightarrow \quad \bigwedge_{k \in K} \alpha_{i,k} = \bigwedge_{i \in I, k \in K} \alpha_{i,k}$$

$$\sum_{k \in K} \alpha_k = \sum_{k \in K} \alpha_{k} \cdot \infty^{(k \in K)} \quad \longleftrightarrow \quad \bigwedge_{k \in K} \alpha_k = \bigwedge_{k \in K} \alpha_k \cdot \infty^{(k \in K)}$$

2.34 Let $K^+ = \{ k \mid \alpha_k \geq 0 \}$ and $K^- = \{ k \mid \alpha_k < 0 \}$. Then if, for example, $n$ is odd, we choose $F_n$ to be $F_{n-1} \cup E_n$, where $E_n \subseteq K^-$ is sufficiently large that $\sum_{k \in E_n} \alpha_k = -\sum_{k \in E_n} (-\alpha_k) < L$

2.35 Goldbach’s sum can be shown to equal

$$\sum_{m,n \geq 2} m^{-n} = \sum_{m \geq 2} \frac{1}{m |m-1|} = 1$$

as follows: By unsumming a geometric series, it equals $\sum_{k \in P, l \geq 1} k^1$; therefore the proof will be complete if we can find a one-to-one correspondence between ordered pairs $(m, n)$ with $m, n \geq 2$ and ordered pairs $(k, l)$ with $k \in P$ and $l \geq 1$, where $m^n = k^l$ when the pairs correspond. If $m \in P$ we let $(m, n) \rightarrow (m^n, 1)$; but if $m \notin P$ we let $(m, n) \rightarrow (1, m^n)$. 

A permutation that consumes terms of one sign faster than those of the other can steer the sum toward any value that it likes.