12. \( \frac{dx}{dt} = (1 + x) \sin y, \quad \frac{dy}{dt} = 1 - x - \cos y \)
13. \( \frac{dx}{dt} = x - y^2, \quad \frac{dy}{dt} = y - x^2 \)
14. \( \frac{dx}{dt} = 1 - xy, \quad \frac{dy}{dt} = x - y^3 \)
15. \( \frac{dx}{dt} = -2x - y - x(x^2 + y^2), \quad \frac{dy}{dt} = x - y + y(x^2 + y^2) \)
16. \( \frac{dx}{dt} = y + x(1 - x^2 - y^2), \quad \frac{dy}{dt} = -x + y(1 - x^2 - y^2) \)

17. Consider the autonomous system

\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x + 2x^3.
\]

(a) Show that the critical point \((0, 0)\) is a saddle point.
(b) Sketch the trajectories for the corresponding linear system by integrating the equation for \(\frac{dy}{dx}\). Show from the parametric form of the solution that the only trajectory on which \(x \to 0, y \to 0\) as \(t \to \infty\) is \(y = -x\).
(c) Determine the trajectories for the nonlinear system by integrating the equation for \(\frac{dy}{dx}\). Sketch the trajectories for the nonlinear system that correspond to \(y = -x\) and \(y = x\) for the linear system.

18. Consider the autonomous system

\[
\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -2y + x^3.
\]

(a) Show that the critical point \((0, 0)\) is a saddle point.
(b) Sketch the trajectories for the corresponding linear system and show that the trajectory for which \(x \to 0, y \to 0\) as \(t \to \infty\) is given by \(x = 0\).
(c) Determine the trajectories for the nonlinear system for \(x \neq 0\) by integrating the equation for \(\frac{dy}{dx}\). Show that the trajectory corresponding to \(x = 0\) for the linear system is unaltered, but that the one corresponding to \(y = 0\) is \(y = x^3/5\). Sketch several of the trajectories for the nonlinear system.

19. The equation of motion of an undamped pendulum is \(\frac{d^2 \theta}{dt^2} + \omega^2 \sin \theta = 0\), where \(\omega^2 = g/L\). Let \(x = \theta, y = d\theta/dt\) to obtain the system of equations

\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\omega^2 \sin x.
\]

(a) Show that the critical points are \((\pm n\pi, 0), n = 0, 1, 2, \ldots\), and that the system is almost linear in the neighborhood of each critical point.
(b) Show that the critical point \((0, 0)\) is a (stable) center of the corresponding linear system. Using Theorem 9.3.2 what can be said about the nonlinear system? The situation is similar at the critical points \((\pm 2n\pi, 0), n = 1, 2, 3, \ldots\). What is the physical interpretation of these critical points?
(c) Show that the critical point \((\pi, 0)\) is an (unstable) saddle point of the corresponding linear system. What conclusion can you draw about the nonlinear system? The situation is similar at the critical points \([\pm (2n - 1)\pi, 0], n = 1, 2, 3, \ldots\). What is the physical interpretation of these critical points?
(d) Choose a value for \(\omega^2\) and plot a few trajectories of the nonlinear system in the neighborhood of the origin. Can you now draw any further conclusion about the nature of the critical point at \((0, 0)\) for the nonlinear system?
(e) Using the value of \(\omega^2\) from part (d) draw a phase portrait for the pendulum. Compare your plot with Figure 9.3.5 for the damped pendulum.

20. (a) By solving the equation for \(\frac{dy}{dx}\), show that the equation of the trajectories of the undamped pendulum of Problem 19 can be written as

\[
\frac{1}{2}y^2 + \omega^2(1 - \cos x) = c, \quad (i)
\]
where \(c\) is a constant of integration.
(b) Multiply Eq. (i) by \(mL^2\). Then express the result in terms of \(\theta\) to obtain

\[
\frac{1}{2}mL^2 \left( \frac{d\theta}{dt} \right)^2 + mgL(1 - \cos \theta) = E, \quad (ii)
\]
where \(E = mL^2c\).