3.21 If \(10^n \leq 2^M < 10^{n+1}\), there are exactly \(n+1\) such powers of 2, because there’s exactly one \(n\)-digit power of 2 for each \(n\). Therefore the answer is \(1 + [M \log 2]\).

Note: The number of powers of 2 with leading digit 1 is more difficult, when \(l > 1\); it’s \(\sum_{0 \leq n \leq M} ([n \log 2 - \log l] - [n \log 2 - \log (l + 1)])\).

3.22 All terms are the same for \(n\) and \(n-1\) except the \(k\)th, where \(n = 2^k q + r\) and \(q\) is odd; we have \(S_n = S_{n-1} + 1\) and \(T_n = T_{n-1} + 2^k q\). Hence \(S_n = n\) and \(T_n = n(n+1)\).

3.23 \(X_n = m \iff \frac{1}{2} m(m-1) < n \leq \frac{1}{2} m(m+1) \iff m^2 - m + \frac{1}{4} < 2n < m^2 + m + \frac{1}{4} \iff m - \frac{1}{2} < \sqrt{2n} < m + \frac{1}{2}\).

3.24 Let \(\beta = \alpha/(\alpha + 1)\). Then the number of times the nonnegative integer \(m\) occurs in \(\text{Spec}(\beta)\) is exactly one more than the number of times it occurs in \(\text{Spec}(\alpha)\). Why? Because \(N(\beta, n) = N(\alpha, n) + n + 1\).

3.25 Continuing the development in the text, if we could find a value of \(m\) such that \(K_m \leq K_n\), we could violate the stated inequality at \(n + 1\) when \(n = 2m + 1\). (Also when \(n = 3m + 1\) and \(n = 3m + 2\).) But the existence of such an \(m = n' + 1\) requires that \(2K_{\lfloor n'/2\rfloor} \leq n'\) or \(3K_{\lfloor n'/3\rfloor} \leq n'\), i.e., that

\[K_{\lfloor n'/2\rfloor} \leq \lfloor n'/2\rfloor \text{ or } K_{\lfloor n'/3\rfloor} \leq \lfloor n'/3\rfloor.\]

Aha. This goes down further and further, implying that \(K_n \leq 0\); but \(K_0 = 1\).

What we really want to prove is that \(K_n\) is strictly greater than \(n\), for all \(n > 0\). In fact, it’s easy to prove this by induction, although it’s a stronger result than the one we couldn’t prove!

(This exercise teaches an important lesson. It’s more an exercise about the nature of induction than about properties of the floor function.)

3.26 Induction, using the stronger hypothesis

\[D_n^{(q)} \leq (q-1) \left(\frac{q}{q-1}\right)^{n+1} - 1, \quad \text{for } n \geq 0.\]

3.27 If \(D_n^{(3)} = 2^m b - a\), where \(b\) is odd and \(a\) is 0 or 1, then \(D_n^{(3)} = 3^m b - a\).

3.28 The key observation is that \(a_n = m^2\) implies \(a_{n+2k+1} = (m+k)^2 + m - k\) and \(a_{n+2k+2} = (m + k)^2 + 2m\), for \(0 \leq k \leq m\); hence \(a_{n+2m+1} = (2m)^2\). The solution can be written in a nice form discovered by Carl Witty:

\[a_{n-1} = 2^l + \left\lfloor \frac{(n-1)^2}{2} \right\rfloor, \quad \text{when } 2^l + 1 \leq n < 2^{l+1} + 1 + 1.\]

3.29 \(D(\alpha', [\alpha n])\) is at most the maximum of the right-hand side of

\[s(\alpha', [n\alpha], \nu') = 2s(\alpha, n, \nu) + S - e - [0 \text{ or } 1] - \nu' + [0 \text{ or } 11].\]