The integral is called the elliptic integral of the first kind. Note that the period depends on the ratio $L/g$ and also the initial displacement $\alpha$ through $k = \sin(\alpha/2)$.

(d) By evaluating the integral in the expression for $T$ obtain values for $T$ that you can compare with the graphical estimates you obtained in Problem 21.

28. A generalization of the damped pendulum equation discussed in the text, or a damped spring–mass system, is the Liénard equation

$$\frac{d^2x}{dt^2} + c(x) \frac{dx}{dt} + g(x) = 0.$$  

If $c(x)$ is a constant and $g(x) = kx$, then this equation has the form of the linear pendulum equation [replace $\sin \theta$ with $\theta$ in Eq. (12) of Section 9.2]; otherwise the damping force $c(x) \frac{dx}{dt}$ and restoring force $g(x)$ are nonlinear. Assume that $c$ is continuously differentiable, $g$ is twice continuously differentiable, and $g(0) = 0$.

(a) Write the Liénard equation as a system of two first order equations by introducing the variable $y = \frac{dx}{dt}$.

(b) Show that $(0, 0)$ is a critical point and that the system is almost linear in the neighborhood of $(0, 0)$.

(c) Show that if $c(0) > 0$ and $g'(0) > 0$, then the critical point is asymptotically stable, and that if $c(0) < 0$ or $g'(0) < 0$, then the critical point is unstable.  

Hint: Use Taylor series to approximate $c$ and $g$ in the neighborhood of $x = 0$.

### 9.4 Competing Species

In this section and the next we explore the application of phase plane analysis to some problems in population dynamics. These problems involve two interacting populations and are extensions of those discussed in Section 2.5, which dealt with a single population. While the equations discussed here are extremely simple, compared to the very complex relationships that exist in nature, it is still possible to acquire some insight into ecological principles from a study of these model problems.

Suppose that in some closed environment there are two similar species competing for a limited food supply; for example, two species of fish in a pond that do not prey on each other, but do compete for the available food. Let $x$ and $y$ be the populations of the two species at time $t$. As discussed in Section 2.5 we assume that the population of each of the species, in the absence of the other, is governed by a logistic equation. Thus

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x), \quad (1a)$$
$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y), \quad (1b)$$

respectively, where $\epsilon_1$ and $\epsilon_2$ are the growth rates of the two populations, and $\epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2$ are their saturation levels. However, when both species are present, each will impinge on the available food supply for the other. In effect, they reduce the growth rates and saturation populations of each other. The simplest expression for reducing the growth rate of species $x$ due to the presence of species $y$ is to replace the growth rate factor $\epsilon_1 - \sigma_1 x$ in Eq. (1a) by $\epsilon_1 - \sigma_1 x - \alpha_1 y$, where $\alpha_1$ is a measure of the degree to