3.36 The sum is

\[
\sum_{k,l,m} 2^{-4^m} [m = \lfloor \log k \rfloor] [l = \lfloor \log k \rfloor] [1 < k < 2^n]
\]

\[
= \sum_{k,l,m} 2^{-14} m [2^m \leq 1 < 2^{m+1}] [2^l \leq k < 2^{l+1}] [0 \leq m < n]
\]

\[
= \sum_{l,m} 4^{-m} [2^m \leq 1 < 2^{m+1}] [0 \leq m < n]
\]

\[
= \sum_{m} 2^{-m} [0 \leq m < n] = 2(1 - 2^{-n}).
\]

3.37 First consider the case \(m < n\), which breaks into subcases based on whether \(m < \frac{1}{2}n\); then show that both sides change in the same way when \(m\) is increased by \(n\).

3.38 At most one \(x_k\) can be noninteger. Discard all integer \(x_k\), and suppose that \(n\) are left. When \(\{x\} \neq 0\), the average of \(\{mx\}\) as \(m \to \infty\) lies between \(\frac{1}{2}\) and \(\frac{1}{4}\); hence \(\{mx_1\} + \ldots + \{mx\} = [\{mx\} + \ldots + mx]\) cannot have average value zero when \(n > 1\).

But the argument just given relies on a difficult theorem about uniform distribution. An elementary proof is possible, sketched here for \(n = 2\): Let \(P_m\) be the point \(\{mx\}, \{my\}\). Divide the unit square \(0 \leq x, y < 1\) into triangular regions \(A\) and \(B\) according as \(x + y < 1\) or \(x + y \geq 1\). We want to show that \(P_m \in B\) for some \(m\), if \(\{x\}\) and \(\{y\}\) are nonzero. If \(P_1 \in B\), we’re done. Otherwise there is a disk \(D\) of radius \(\epsilon > 0\) centered at \(P_1\) such that \(D \subseteq A\). By Dirichlet’s box principle, the sequence \(P_1, \ldots, P_N\) must contain two points with \(|P_k - P_j| < \epsilon\) and \(k > j\), if \(N\) is large enough.

![Diagram](image)

It follows that \(P_{k-j}\) is within \(\epsilon\) of \((1,1) - P_1\); hence \(P_{k-j} \in B\).

3.39 Replace \(j\) by \(b = j\) and add the term \(j = 0\) to the sum, so that exercise 15 can be used for the sum on \(j\). The result,

\[
[x/b^k] \ [x/b^{k+1}] + b = 1,
\]

telescopes when summed on \(k\).