500 ANSWERS TO EXERCISES

3.50 H. S. Wilf observes that the functional equation \( f(x^2 - 1) = f(x)^2 \) would determine \( f(x) \) for all \( x \geq \phi \) if we knew \( f(x) \) on any interval \( (\phi, \phi + \epsilon) \).

3.51 There are infinitely many ways to partition the positive integers into three or more generalized spectra with irrational \( \alpha_k \); for example,

\[
\text{Spec}(2\alpha; 0) \cup \text{Spec}(4\alpha; -\alpha) \cup \text{Spec}(4\alpha; -3\alpha) \cup \text{Spec}(\beta; 0)
\]

works. But there’s a precise sense in which all such partitions arise by “expanding” a basic one, \( \text{Spec}(\alpha) \cup \text{Spec}(\beta) \); see [128]. The only known rational examples, e.g.,

\[
\text{Spec}(7; -3) \cup \text{Spec}(7/2; -1) \cup \text{Spec}(7/2; 0)
\]

are based on parameters like those in the stated conjecture, which is due to A. S. Fraenkel [103].

3.52 Partial results are discussed in [77, pages 30–31].

4.1 1, 2, 4, 6, 16, 12.

4.2 Note that \( m_p + n_p = \min(m_p, n_p) + \max(m_p, n_p) \). The recurrence \( \text{lcm}(m, n) = (n/(n \mod m)) \text{lcm}(n \mod m, m) \) is valid but not really advisable for computing \( \text{lcm} \)'s; the best way known to compute \( \text{lcm}(m, n) \) is to compute \( \text{gcd}(m,n) \) first and then to divide \( mn \) by the \( \text{gcd} \).

4.3 This holds if \( x \) is an integer, but \( n(x) \) is defined for all real \( x \). The correct formula,

\[
n(x) = \pi(x - 1) = \lfloor x \rfloor \text{ is prime}\n\]

is easy to verify.

4.4 Between \( 1/\phi \) and \( 0 \), we’d have a left-right reflected Stern-Brocot tree with all denominators negated, etc. So the result is \( \text{all} \) fractions \( m/n \) with \( m \perp n \). The condition \( m'n' - mn = 1 \) still holds throughout the construction. (This is called the Stern-Brocot wreath, because we can conveniently regard the final \( 1/1 \) as identical to the first \( 1/\phi \), thereby joining the trees in a cycle at the top. The Stern-Brocot wreath has interesting applications to computer graphics because it represents all rational directions in the plane.)

4.5 \( L^k = (1^k) \) and \( R^k = (1^0) \); this holds even when \( k < 0 \). (We will find a general formula for any product of \( L \)'s and \( R \)'s in Chapter 6.)

4.6 \( a = b \). (Chapter 3 defined \( x \mod 0 = x \), primarily so that this would be true.)

4.7 We need \( m \mod 10 = 0 \), \( m \mod 9 = k \), and \( m \mod 8 = 1 \). But \( m \) can’t be both even and odd.