A ANSWERS TO EXERCISES 501

4.8 We want \(10x + 6y \equiv 1 \mod 15\); hence \(5y \equiv 0 \mod 3\); hence \(y \equiv 0 \mod 3\). We must have \(x = 0 \) or \(1\).

4.9 \(32^{k+1} \mod 4 \cdot 3 = 3\), so \((3^{2k+1} - 1)/2\) is odd. The stated number is divisible by \((3^7 - 1)/2\) and \((3^7 - 1)/2\) (and by other numbers).

4.10 \(999(1 - \frac{1}{3})(1 - \frac{1}{3}) = 648\).

4.11 \(\sigma(0) = 1;\ \sigma(1) = 1;\ \sigma(n) = 0\) for \(n > 1\). (Generalized Mobius functions defined on arbitrary partially ordered structures have interesting and important properties, first explored by Weisner [299] and developed by many other people, notably Gian-Carlo Rota [254].)

4.12 \(\sum d \cdot \sum k \cdot \mu(d/k) g(k) = \sum k \cdot \mu(k) g(k) = \sum m/k g(m)\), by (4.7) and (4.9).

4.13 (a) \(n_p \leq 1\) for all \(p\); (b) \(\mu(n) \neq 0\).

4.14 True when \(k > 0\). Use (4.12), (4.14), and (4.15).

4.15 No. For example, \(e_5 \mod 5 = 2\) or \(3\); \(e_5 \mod 11 = 2, 3, 7, \) or \(10\).

4.16 \(1/e_1 + 1/e_2 + \ldots + 1/e_n = 1 - 1/(e_n(e_n - 1)) = 1-1/(e_n+1-1)\).

4.17 We have \(f_n \mod f_m = 2\); hence \(gcd(f_n, f_m) = \gcd(2, f_m) = 1\). (Incidentally, the relation \(f_n = f_0 f_1 \ldots f_{n-1} + 2\) is very similar to the recurrence that defines the Euclid numbers \(e_n\).)

4.18 If \(n = \frac{q}{m}\) and \(m\) is odd, \(2^n + 1 = (2^m + 1)(2^n-2^n-\ldots-2^m-1)\).

4.19 Let \(p_1 = 2\) and let \(p_n\) be the smallest prime greater than \(2^{p_{n-1}}\). Then \(2^{p_{n-1}} < p_n < 2^{p_{n-1} + 1}\), and it follows that we can take \(b = \lim_{n \to \infty} \frac{\lg |n|}{p_n}\) where \(\lg |n|\) is the function \(\lg\) iterated \(n\) times. The stated numerical value comes from \(p_2 = 5\), \(p_3 = 37\). It turns out that \(p_4 = 237 + 9\), and this gives the more precise value

\[
b \approx \ 1.2516475977905
\]

(but no clue about \(p_5\)).

4.20 By Bertrand’s postulate, \(P_n < 10^n\). Let

\[K = \sum_{k \geq 1} 10^{-k} p_k = .20300805\ldots\]

Then \(10^n K \equiv P_n + \text{fraction} \mod 10^{2n-1}\).

4.21 The first sum is \(n(n)\), since the summand is \((k + 1)\) is prime). The inner sum in the second is \(\sum_{1 \leq k < m} |k \cdot m|\), so it is greater than \(1\) if and only if \(m\) is composite; again we get \(n(n)\). Finally \([\{m/n\}] = [n/1]\), so the third sum is an application of Wilson’s theorem. To evaluate \(n(n)\) by any of these formulas is, of course, sheer lunacy.