4.22 \( \frac{(b^m - 1)}{(b - 1)} = \left( \frac{(b^m - 1)}{(b - 1)} \right) \left( \frac{b^{m-n} - 1}{b - 1} \right) + \cdots + 1 \). [The only prime numbers of the form \((10^p - 1)/9\) for \(p < 2000\) occur when \(p = 2, 19, 23, 317, 1031\).]

4.23 \( \rho(2k+1) = 0; \rho(2k) = \rho(k) + 1 \), for \(k \geq 1\). By induction we can show that \(\rho(n) = \rho(n - 2^m)\), if \(n > 2^m\) and \(m > \rho(n)\). The kth Hanoi move is disk \(p(k)\), if we number the disks 0, 1, ..., \(n - 1\). This is clear if \(k\) is a power of 2. And if \(2^m < k < 2^{m+1}\), we have \(p(k) < m\); moves \(k\) and \(k - 2^m\) correspond in the sequence that transfers \(m + 1\) disks in \(T_{m+1} + T_m\) steps.

4.24 The digit that contributes \(dp^m\) to \(n\) contributes \(dp^{m-1} + \cdots + d = d(p^m - 1)/(p - 1)\) to \(e_{p}(n!)\), hence \(e_{p}(n!) = (n - \gamma_p(n))/(p - 1)\).

4.25 \(m \mod n \iff m_p = 0\) or \(m_p = n_p\), for all \(p\). It follows that (a) is true. But (b) fails, in our favorite example \(m = 12, n = 18\). (This is a common fallacy.)

4.26 Yes, since \(\mathcal{S}_N\) defines a subtree of the Stern-Brocot tree.

4.27 Extend the shorter string with M’s (since M lies alphabetically between L and R) until both strings are the same length, then use dictionary order. For example, the topmost levels of the tree are LL < LM < LR < MM < RL < RM < RR. (Another solution is to append the infinite string RL" to both inputs, and to keep comparing until finding L < R.)

4.28 We need to use only the first part of the representation:

```
    L L L L L L L L R R R R R R R R R R
    1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
```

The fraction \(\frac{2}{3}\) appears because it’s a better upper bound than \(\frac{1}{2}\), not because it’s closer than \(\frac{3}{4}\). Similarly, \(\frac{25}{36}\) is a better lower bound than \(\frac{22}{36}\). The simplest upper bounds and the simplest lower bounds all appear, but the next really good approximation doesn’t occur until just before the string of R’s switches back to L.

4.29 \(1/x\). To get \(1 - x\) from \(x\) in binary notation, we interchange 0 and 1; to get \(1/a\) from \(a\) in Stern-Brocot notation, we interchange L and R. (The finite cases must also be considered, but they must work since the correspondence is order preserving.)

4.30 The \(m\) integers \(x \in [A, A+m)\) are different \(\mod m\); hence their residues \(x \mod m_1, \ldots, x \mod m_m\) run through all \(m_1 \ldots m_m = m\) possible values, one of which must be \((a_1 \mod m_1, \ldots, a_t \mod m_t)\) by the pigeonhole principle.

4.31 A number in radix \(b\) notation is divisible by \(d\) if and only if the sum of its digits is divisible by \(d\), whenever \(b \equiv 1 \pmod{d}\). This follows because \((a_\ldots a_0)_b = a_m b^m + \cdots + a_0 b^0 \equiv a_m + \cdots + a_0\).