4.32 The \( \varphi(m) \) numbers \( \{ kn \mod m \ k \perp m \text{ and } 0 \leq k < m \} \) are the numbers \( \{ k \ k \perp m \text{ and } 0 \leq k < m \} \) in some order. Multiply them together and divide by \( \prod_{0 \leq k < m, \ k \perp m} k \).

4.33 Obviously \( h(1) = 1 \). If \( m \perp n \) then \( h(mn) = \sum_{d \mid mn} f(d) g(mn/d) = \sum_{c \mid m, d \mid n} f(c) g((m/c)(n/d)) = \sum_{c \mid m} \sum_{d \mid n} f(c) g(m/c) f(d) g(n/d) \); this is \( h(m) h(n) \), since \( c \mid 1 \) \( d \) for every term in the sum.

4.34 \( g(m) = \sum_{d \mid m} f(d) = \sum_{d \mid m} f(m/d) = \sum_{d \mid m} f(c) \) if \( f(x) \) is zero when \( x \) is not an integer.

4.35 The base cases are
\[
I(0,n) = 0; \quad I(m,0) = 1.
\]
When \( m, n > 0 \), there are two rules, where the first is trivial if \( m > n \) and the second is trivial if \( m < n \):
\[
I(m,n) = I(m, n \mod m) \quad \left[ \frac{n}{m} \right] I(n \mod m, m); \\
I(m,n) = I(m \mod n, n),
\]

4.36 A factorization of any of the given quantities into nonunits must have \( m^2 \) \( 10n^2 = \pm 2 \) or \( \pm 3 \), but this is impossible \( \mod 10 \).

4.37 Let \( a_n = 2^{-n} \ln(e_n - \frac{1}{2}) \) and \( b_n = 2^{-n} \ln(e_n + \frac{1}{2}) \). Then
\[
e_n = \frac{E^2}{2} + \frac{1}{2} \iff a_n \leq \ln E < b_n.
\]
And \( a_{n-1} < a_n < b_n < b_{n+1} \), so we can take \( E = \lim_{n \to \infty} e^{a_n} \). In fact, it turns out that
\[
E^2 = \frac{3}{2} \prod_{n=1}^{\infty} \left( 1 + \frac{1}{(2e_n - 1)^2} \right)^{1/2^n},
\]
a product that converges rapidly to \( \left( 1.26408473530530111 \right)^2 \). But these observations don’t tell us what \( e_n \) is, unless we can find another expression for \( E \) that doesn’t depend on Euclid numbers.

4.38 \( a^n - b^n = (a^n - b^m)(a^{n-m}b^m + a^{n-2m}b^m + \ldots + a^m b^{n-m} + b^{n-m}) + b^m \). \( a \) \( b \) are perfect squares, so is
\[
a_1 a_1 b_1 \ldots b_n / c_1^2 \ldots c_r^2,
\]
where \( \{ a_1, \ldots, a_t \} \cap \{ b_1, \ldots, b_u \} = \{ c_1, \ldots, c_r \} \). (It can be shown, in fact, that the sequence \( \{ S(1), S(2), S(3), \ldots \} \) contains every nonprime positive integer exactly once.)