4.40 Let \( f(n) = \prod_{1 \leq k \leq n, p \mid k} k^{(n/p)} \cdot \frac{n!}{p^{\lfloor n/p \rfloor}} \) and \( g(n) = \frac{n!}{p^{\epsilon_p(n)}} \). Then
\[
g(n) = f(n)f([n/p])f\left([n/p^2]\right)\ldots = f(n)g\left([n/p]\right).
\]
Also \( f(n) = a_0!\lfloor n/p \rfloor \equiv a_0!(-1)^{[n/p]} \pmod p \), and \( \epsilon_p(n!) = [n/p] + \epsilon_p\left([n/p]\right) \). These recurrences make it easy to prove the result by induction.
(Several other solutions are possible.)

4.41 (a) If \( n^2 \equiv -1 \pmod p \) then \( n^{(p-1)/2} \equiv -1 \); but Fermat says it’s +1. (b) Let \( n = ((p - 1)/2)! \); we have \( n \equiv (-1)^{(p-1)/2} \prod_{1 \leq k < p/2} (p-k) = (p-1)!/n \), hence \( n^2 \equiv (p-1)! \).

4.42 First we observe that \( k | l \iff k | 1 + ak \) for any integer \( a \), since \( \gcd(k, l) = \gcd(k, 1+ak) \) by Euclid’s algorithm. Now
\[
m \perp l \text{ and } n \perp n \implies m \perp l \text{ and } n \perp n
\]
Similarly
\[
m \perp n \text{ and } n \perp n \iff mn' + nm' \perp n'.
\]
Hence
\[
m \perp n \text{ and } m' \perp n' \text{ and } n \perp n' \iff mn' + nm' \perp n n'.
\]

4.43 We want to multiply by \( L^{-1}R \), then by \( R^{-1}L^{-1}RL \), then \( L^{-1}R \), then \( R^{-2}L^{-1}RL^2 \), etc.; the \( n \)th multiplier is \( R^{\lfloor n/p \rfloor}L^{-1}RL^{\lfloor n/p \rfloor} \), since we must cancel \( p(n) \) R’s. And \( R^{-m}L^{1}RL^{m} = R^{0}(-1)^{1} \).

4.44 We can find the simplest rational number that lies in
\[
[.3155, .3165] = [\frac{631}{2000}, \frac{633}{2000}]
\]
by looking at the Stern-Brocot representations of \( \frac{631}{2000} \) and \( \frac{633}{2000} \) and stopping just before the former has \( R \) where the latter has \( L 
\):
\[
(m_1, n_1, m_2, n_2) := (631, 2000, 633, 2000);
\text{while} \ m_1 > n_1 \text{ or } m_2 < n_2 \text{ do}
\text{if} \ m_2 < n_2 \text{ then} \ (\text{output}(L); \ [n_1, n_2] := [n_1, n_2] - (m_1, m_2) \)
\text{else} \ (\text{output}(R); \ [m_1, m_2] := [m_1, m_2] - (n_1, n_2) \).
\]
The output is LLLRRRRR = \( \frac{6}{19} \approx .3158 \). Incidentally, an average of .334 implies at least 287 at bats.