4.45 \( x^2 \equiv x \pmod{10^n} \iff x(x-1) \equiv 0 \pmod{2^n} \) and \( x(x-1) \equiv 0 \pmod{5^n} \iff x \equiv 0 \pmod{2^n} \). The last step is justified because \( x(x-1) \pmod{5} = 0 \) implies that either \( x \) or \( x-1 \) is a multiple of 5, in which case the other factor is relatively prime to \( 5^n \) and can be divided from the congruence.

So there are at most four solutions, of which two \( (x = 0 \text{ and } x = 1) \) don’t qualify for the title “n-digit number” unless \( n = 1 \). The other two solutions have the forms \( x \) and \( 10001 - x \), and at least one of these numbers is \( \geq 100 \). When \( n = 4 \) the other solution, 10001 - 9376 = 625, is not a four-digit number. We expect to get two n-digit solutions for about 90% of all \( n \), but this conjecture has not been proved.

(Such self-reproducing numbers have been called “automorphic.”)

4.46 (a) If \( j \mid k \iff k = \gcd(j,k) \), we have \( n^{k/k} \cdot \gcd(i,k) = n^i \equiv 1 \) and \( n^{k/k} \equiv 1 \). (b) Let \( n = pq \), where \( p \) is the smallest prime divisor of \( n \). If \( 2^n \equiv 1 \pmod{n} \) then \( 2^n \equiv 1 \pmod{p} \). Also \( 2^{p-1} \equiv 1 \pmod{p} \); hence \( 2^{\gcd(p-1,n)} \equiv 1 \pmod{p} \). But \( \gcd(p-1,n) = 1 \) by the definition of \( p \).

4.47 If \( n^{m-1} \equiv 1 \pmod{m} \) we must have \( n \not\equiv 1 \pmod{m} \). If \( n^k \equiv n^j \) for some \( 1 \leq j < k < m \), then \( n^{k-j} \equiv 1 \) because we can divide by \( n^j \). Therefore if the numbers \( n^1 \pmod{m}, \ldots, n^{n-1} \pmod{m} \) are not distinct, there is a \( k \leq m-1 \) with \( n^k \equiv 1 \). The least such \( k \) divides \( m-1 \), by exercise 46(a). But then \( kq = (m-1)/p \) for some prime \( p \) and some positive integer \( q \); this is impossible, since \( n^{kq} \not\equiv 1 \). Therefore the numbers \( n^1 \pmod{m}, \ldots, n^{m-1} \pmod{m} \) are distinct and relatively prime to \( m \). Therefore the numbers \( 1, \ldots, m-1 \) are relatively prime to \( n \), and \( m \) must be prime.

4.48 By pairing numbers up with their inverses, we can reduce the product \( n^2 \equiv 1 \pmod{m} \) to \( \prod_{1 \leq i < n, n^i \equiv 1 \pmod{m}} n \). Now we can use our knowledge of the solutions to \( n^2 \equiv 1 \pmod{m} \). By residue arithmetic we find that the result is \( m = 1 \) if \( m = 4 \), \( p^k \), or \( 2p^k \) \( (p > 2) \); otherwise it’s +1.

4.49 (a) Either \( m < n \) (\( \Phi(N-1) \) cases) or \( m = n \) (one case) or \( m > n \) (\( \Phi(N-1) \) again). Hence \( R(N) = 2\Phi(N-1) + 1 \). (b) \( \Phi(N) = 2\sum_{d \mid N} \mu(d)[N/d][N/d-1] \); hence the stated result holds if and only if

\[
\sum_{d \mid N} \mu(d)[N/d][N/d-1] = 1 \quad \text{for } N \geq 1
\]

And this is a special case of (4.61) if we set \( f(x) = (x \geq 1) \).