• We also have Boolean variables \( B_{ij} \), that are true iff square \([i, j]\) is breezy; we include these variables only for the observed squares—in this case, \([1,1]\), \([1,2]\), and \([2,1]\).

The next step is to specify the full joint distribution, \( P(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1}) \). Applying the product rule, we have

\[
P(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1}) = P(P_{1,1}) \cdots P(P_{4,4}) P(B_{1,1}) \cdots P(B_{2,1})
\]

This decomposition makes it easy to see what the joint probability values should be. The first term is the conditional probability distribution of a breeze configuration, given a pit configuration; its values are 1 if the breezes are adjacent to the pits and 0 otherwise. The second term is the prior probability of a pit configuration. Each square contains a pit with probability 0.2, independently of the other squares; hence,

\[
P(P_{1,1}) = 0.2^n \times 0.8^{4-n}
\]

For a particular configuration with exactly \( n \) pits, \( P(P_{1,1}, \ldots, P_{4,4}) = 0.2^n \times 0.8^{4-n} \).

In the situation in Figure 13.5(a), the evidence consists of the observed breeze (or its absence) in each square that is visited, combined with the fact that each such square contains no pit. We abbreviate these facts as \( b = A_{1,2} A_{2,1} \) and \( known = \neg P_{1,1} A \neg P_{1,2} A P_{2,1} \). We are interested in answering queries such as \( P(P_{1,3} \text{ known}, b) \): how likely is it that \([1,3]\) contains a pit, given the observations so far?

To answer this query, we can follow the standard approach of Equation (13.9), namely, summing over entries from the full joint distribution Let \( Unknown \) be the set of \( P_{i,1} \) vari-

Figure 13.5 (a) After finding a breeze in both \([1,2]\) and \([2,1]\), the agent is stuck—there is no safe place to explore. (b) Division of the squares into Known, Frontier, and Other, for a query about \([1,3]\).
ables for squares other than the Known squares and the query square [1,3]. Then, by Equation (13.9), we have
\[ P(P_{1,3} | \text{known}, b) = \sum_{\text{unknown}} P(P_{1,3}, \text{unknown}, \text{known}, b). \]
The full joint probabilities have already been specified, so we are done—that is, unless we care about computation. There are 12 unknown squares; hence the summation contains \(2^{12} = 4194304 \) terms. In general, the summation grows exponentially with the number of squares.

Surely, one might ask, aren’t the other squares irrelevant? How could [4,4] affect whether [1,3] has a pit? Indeed, this intuition is correct. Let \( \text{Frontier} \) be the pit variables (other than the query variable) that are adjacent to visited squares, in this case just [2,2] and [3,3]. Also, let \( \text{Other} \) be the pit variables for the other unknown squares; in this case, there are 10 other squares, as shown in Figure 13.5(b). The key insight is that the observed breezes are conditionally independent of the other variables, given the known, frontier, and query variables. To use the insight, we manipulate the query formula into a form in which the breezes are conditioned on all the other variables, and then we apply conditional independence:

\[ P(P_{1,3} | \text{known}, b) = \sum_{\text{unknown}} P(P_{1,3}, \text{known}, \text{unknown}) = \sum_{\text{unknown}} P(b | \text{known}, P_{1,3}, \text{unknown}, \text{known}, \text{unknown}) P(P_{1,3}, \text{known}, \text{unknown}). \]

(by the product rule)

where the final step uses conditional independence: \( b \) is independent of \( \text{Other} \) given \( \text{Known}, P_{1,3}, \) and \( \text{Frontier} \). Now, the first term in this expression does not depend on the \( \text{Other} \) variables, so we can move the summation inward:

\[ P(P_{1,3} | \text{known}, b) = \sum_{\text{known}} P(b | \text{known}, P_{1,3}, \text{frontier}, \text{other}) \sum_{\text{other}} P(P_{1,3}, \text{known}, \text{frontier}, \text{other}). \]

By independence, as in Equation (13.20), the prior term can be factored, and then the terms can be reordered:

\[ P(P_{1,3} | \text{known}, b) = \sum_{\text{frontier}} P(b | \text{known}, P_{1,3}, \text{frontier}) \sum_{\text{other}} P(P_{1,3}, \text{known}, \text{frontier}, \text{other}) \sum_{\text{other}} P(\text{other}). \]

where the final step uses conditional independence: \( b \) is independent of \( \text{Other} \) given \( \text{Known}, P_{1,3}, \) and \( \text{Frontier} \). Now, the first term in this expression does not depend on the \( \text{Other} \) variables, so we can move the summation inward: