this is divisible by \(n\).

Therefore the answer is: Either \(n = 1\) or \(n \neq 4\) is composite.

4.54 \(\varepsilon_2(1 \cdot 000!) > 500\) and \(\varepsilon_3(1 \cdot 000!) = 249\), hence \(1 \cdot 000! = a \cdot 10^{249}\) for some even integer \(a\). Since \(1000 = (1300)_3\), exercise 40 tells us that \(a \cdot 2^{249} = 1000! \mod 249 \equiv 1\) (mod 5). Also \(2^{249} \equiv 2\), hence \(a \equiv 2\), hence a mod 10 = 2 or 7; hence the answer is \(2 \cdot 10^{249}\).

4.55 One way is to prove by induction that \(P_{2n}/P_n(n + 1)\) is an integer; this stronger result helps the induction go through. Another way is based on showing that each prime \(p\) divides the numerator at least as often as it divides the denominator. This reduces to proving the inequality

\[
\sum_{k=1}^{n} \frac{k}{m} \geq 4 \sum_{k=1}^{n} \frac{k}{m},
\]

which follows from

\[
\left\lfloor \frac{2n - 1}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor \geq \left\lfloor \frac{n}{m} \right\rfloor
\]

The latter is true when \(0 \leq n < m\), and both sides increase by 4 when \(n\) is increased by \(m\).

4.56 Let \(f(n) = \sum_{k=1}^{n} \min(k, 2n-k) [m/k], \quad g(n) = \sum_{k=1}^{n} (2n - 2k + 1) \times \left[\frac{m}{2k + 1}\right].\) The number of times \(p\) divides the numerator of the stated product is \(f(p) + f(p^2) + f(p^3) + \cdots\), and the number of times \(p\) divides the denominator is \(g(p) + g(p^2) + g(p^3) + \cdots\). But \(f(n) = g(n)\) whenever \(m\) is odd, by exercise 2.32. The stated product therefore reduces to \(2^{\frac{n(n+1)}{2}}\), by exercise 3.22.

4.57 The hint suggests a standard interchange of summation, since

\[
\sum_{1 \leq m \leq n} \left\lfloor \frac{d}{m} \right\rfloor = \sum_{0 < k \leq n/d} d k = \left\lfloor \frac{n}{d} \right\rfloor.
\]

Calling the hinted sum \(X(n)\), we have

\[
\Sigma(m + n) - \Sigma(m) - \Sigma(n) = \sum_{d \in S(m,n)} \varphi(d).
\]

On the other hand, we know from (4.54) that \(\Sigma(n) = \frac{1}{2}n(n + 1)\). Hence \(\Sigma(m + n) - X(m) - \Sigma(n) = mn\).

4.58 The function \(f(m)\) is multiplicative, and when \(m = p^k\) it equals \(1 + p + \cdots + p^k\). This is a power of 2 if and only if \(p\) is a Mersenne prime and \(k = 1\). For \(k\) must be odd, and in that case the sum is

\[
(1 + p)(1 + p^2 + p^4 + \cdots + p^{k-1})
\]