this is divisible by $n$.

Therefore the answer is: Either $n = 1$ or $n \neq 4$ is composite.

4.54 $e_2(1000!) > 500$ and $e_3(1000!) = 249$, hence $1000! = q \cdot 10^{249}$ for some even integer $a$. Since $1000 = (1300)_5$, exercise 40 tells us that $a \cdot 2^{249} = 1000!/5^{249} \equiv 1 \pmod{5}$. Also $2^{249} \equiv 2$, hence $a \equiv 2$, hence a mod 10 = 2 or 7; hence the answer is 2.10^{249}.

4.55 One way is to prove by induction that $P_{2n}/P_n^4(n + 1)$ is an integer; this stronger result helps the induction go through. Another way is based on showing that each prime $p$ divides the numerator at least as often as it divides the denominator. This reduces to proving the inequality

$$\sum_{k=1}^{2n} \left\lfloor \frac{k}{m} \right\rfloor \geq 4 \sum_{k=1}^{n} \left\lfloor \frac{k}{m} \right\rfloor,$$

which follows from

$$\left\lfloor \frac{(2n-1)\cdot m}{m} \right\rfloor + \left\lfloor \frac{2n\cdot m}{m} \right\rfloor \geq \left\lfloor \frac{n}{m} \right\rfloor.$$

The latter is true when $0 \leq n < m$, and both sides increase by 4 when $n$ is increased by $m$.

4.56 Let

$$f(m) = \sum_{k=1}^{2n-1} \min(k, 2n-k)[m,k],$$

and

$$g(m) = \sum_{k=1}^{n-1} (2n-2k-1) x \left[m/(2k+1)\right].$$

The number of times $p$ divides the numerator of the stated product is $f(p) + f(p^2) + f(p^3) + \cdots$, and the number of times $p$ divides the denominator is $g(p) + g(p^2) + g(p^3) + \cdots$. But $f(m) = g(m)$ whenever $m$ is odd, by exercise 2.32. The stated product therefore reduces to $2^{n(n+1)}$, by exercise 3.22.

4.57 The hint suggests a standard interchange of summation, since

$$\sum_{1 \leq m \leq n} [d|m] = \sum_{0<k<n/d} \sum_{m \equiv dk \equiv [n/d]} m = [n/d].$$

Calling the hinted sum $X(n)$, we have

$$\Sigma(m + n) - \Sigma(m) - \Sigma(n) = \sum_{d\in S(m,n)} \varphi(d).$$

On the other hand, we know from (4.54) that $\Sigma(n) = \frac{1}{2}n(n + 1)$. Hence

$$\Sigma(m + n) - X(m) - \Sigma(n) = mn.$$

4.58 The function $f(m)$ is multiplicative, and when $m = p^k$ it equals $1 + p + \cdots + p^k$. This is a power of 2 if and only if $p$ is a Mersenne prime and $k = 1$. For $k$ must be odd, and in that case the sum is

$$(1 + p)(1 + p^2 + p^4 + \cdots + p^{k-1})$$