and $(k-1)/2$ must be odd, etc. The necessary and sufficient condition is that $m$ be a product of distinct Nersenne primes.

4.59 Proof of the hint: If $n = 1$ we have $x_1 = a = 2$, so there’s no problem. If $n > 1$ we can assume that $x_1 \leq \ldots \leq x_n$. Case 1: $x_1 \ldots + x_{n-1} + (x_n - 1)^{-1} \geq 1$ and $x_n > x_{n-1}$. Then we can find $\beta \geq x_n - 1 \geq x_{n-1}$ such that $x_1 \ldots + x_{n-1} + \beta^{-1} = 1$; hence $x_n \leq \beta + 1 \leq e_n$ and $x_1 \ldots x_n \leq x_1 \ldots x_{n-1} (\beta + 1) \leq e_1 \ldots e_n$, by induction. There is a positive integer $m$ such that $a = x_1 \ldots x_n / m$; hence $a \leq e_1 \ldots e_n = e_{n+1}$. Case 2: $x_1 \ldots x_{n-1} (\alpha + 1) \leq e_1 \ldots e_{n-1}$, and we have $x_1 \ldots x_n (\alpha + 1) \leq e_1 \ldots e_n$. Case 3: $x_1 \ldots + x_{n-1} + (x_n - 1)^{-1} \geq 1$ and $x_n = x_{n-1}$. Let $a = x_n$ and $a^{-1} + (a - 1)^{-1} = (a - 2)^{-1} + \zeta^{-1}$. Then we can show that $a 3 4$ and $(a-2)(a+1) \geq a^2$. So there’s $a \geq \zeta$ such that $x_1 \ldots + x_{n-2} + (a - 2)^{-1} + \beta^{-1} = 1$; it follows by induction that $x_1 \ldots x_n \leq x_1 \ldots x_{n-2}(a-2)(\beta+1) \leq e_1 \ldots e_n$, and we can finish as before. Case 3: $x_1 \ldots + x_{n-1} + (x_n - 1)^{-1} < 1$.

4.60 The main point is that $\theta < \frac{3}{2}$. Then we can take $p_1$ sufficiently large (to meet the conditions below) and $p_n$ to be the least prime greater than $p_{n-1}$. With this definition let $a_n = 3^{-n} \ln p_n$ and $b_n = 3^{-n} \ln (p_n + 1)$. If we can show that $a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$, we can take $P = \lim_{n \to \infty} e^{a_n}$ as in exercise 37. But this hypothesis is equivalent to $p_{n-1}^3 \leq p_n < (p_{n-1} + 1)^3$. If there’s no prime $p_n$ in this range, there must be a prime $p < p_{n-1}^3$ such that $p + cp^\theta > (p_{n-1} + 1)^3$. But this implies that $cp^\theta > 3p^{2/3}$, which is impossible when $p$ is sufficiently large.

We can almost certainly take $p_1 = 2$, since all available evidence indicates that the known bounds on gaps between primes are much weaker than the truth (see exercise 69). Then $p_2 = 11$, $p_3 = 1361$, $p_4 = 2521008887$, and $1.3067788383 < P < 1.30677883869$.

4.61 Let $\tilde{m}$ and $\tilde{n}$ be the right-hand sides; observe that \[ m' \tilde{n} = 1, \] hence $m' \tilde{n} \not\mid \tilde{n}$. Also $m' \tilde{n} > m' / \tilde{n}'$.\, So $n \geq \tilde{n} > (n + N) / n' - 1 \geq 0$. So we have $m' / \tilde{n} \geq m' / n'$. If equality doesn’t hold, we have \[ n'' = (m' n' - m' \tilde{n}) n'' = n' (m' \tilde{n} - m' n') + \tilde{n} (m'' n' - m'' \tilde{n}) \] \[ \geq n' + \tilde{n} > N, \] a contradiction.