where the parameter $\beta$, in the range of 0 to 1, can be interpreted as the faith we have on the response model relative to the data model. As $\beta$ decreases, the effect of the response model decreases correspondingly. When $\beta = 0$, the RAPMF collapses to PMF.

More importantly, the expectations of Bernoulli distributions, $\mu_k$’s should be in the range of 0 to 1. With the performance consideration, the logistic function is usually adopted to constrain the range of $\mu_k$’s [15],

$$g(\mu_k) = \frac{1}{1 + \exp(-\mu_k)}, \quad k = 1, \ldots, D. \quad (13)$$

Similarly, we place a zero mean Gaussian prior on $\mu_k$ to regularize it.

Note that in Eq. (10), we use only one parameter for each possible rating value, so all the users and items share the same probability as long as the rating values are the same. This is a simple approach to capture the intuition that the rating assigned to an item may influence the chance that it got rated. This motivates us to name this model as rating dominant response model. We refer to PMF with Rating dominate response model as RAPMF-r.

By incorporating the response model in Eq. (12) and the PFM model in Eq. (6) into RAPMF in Eq. (8), we obtain the log-likelihood of the RAPMF-r as follows:

$$\mathcal{L}(U, V, \sigma^2, \mu) = \beta \sum_{i=1}^{N} \sum_{j=1}^{M} \log \left( \sum_{k=1}^{D} \alpha_{kij} \mathcal{N}(k|U^T V, \sigma^2) \right) - \frac{1}{2\sigma_u^2} \|\mu\|^2 - \frac{1}{2\sigma_u^2} \|U\|^2 - \frac{1}{2\sigma_v^2} \|V\|^2 + C,$$

$$\quad \sum_{(i,j,x) \in Q} \left( x_{ij} - \frac{U_i^T V_j}{\sigma^2} \right)^2 - \frac{1}{2\sigma_u^2} \|U\|^2 - \frac{1}{2\sigma_v^2} \|V\|^2 + C, \quad (14)$$

where $C$ denotes the constant terms and $\alpha_{kij}$ is defined as

$$\alpha_{kij} = (g(\mu_k)^{[r_{ij}=1]}(1 - g(\mu_k))^{[r_{ij}=0]}). \quad (15)$$

The gradient of $\mathcal{L}$ with respect to $U_i$ is:

$$\frac{\partial \mathcal{L}}{\partial U_i} = -\beta \sum_{j=1}^{M} \sum_{k=1}^{D} \alpha_{kij} \mathcal{N}(k|U^T V, \sigma^2) \frac{U_i^T V_j - k}{\sigma^2} V_j - \sum_{j=1}^{M} (U_i^T V_j - x_{ij})[r_{ij} = 1] V_j - \lambda_u U_i. \quad (16)$$

Similarly, the gradient of $\mathcal{L}$ with respect to $V_j$ is:

$$\frac{\partial \mathcal{L}}{\partial V_j} = -\beta \sum_{i=1}^{N} \sum_{k=1}^{D} \alpha_{kij} \mathcal{N}(k|U^T V, \sigma^2) \frac{U_i^T V_j - k}{\sigma^2} U_i - \sum_{j=1}^{M} (U_i^T V_j - x_{ij})[r_{ij} = 1] U_i - \lambda_v V_j. \quad (17)$$

Both Eq. (16) and Eq. (17) consist of three terms. The first term corresponds to the change due to the response model, the second term is the change due to the data model and third is a regularization to avoid overfitting. Note that by adjusting $\beta$, we effectively alter the weight of the response model when updating parameters.

Finally, the gradient of $\mathcal{L}$ with respect to $\mu_i$ is

$$\frac{\partial \mathcal{L}}{\partial \mu_i} = \sum_{j=1}^{M} \sum_{k=1}^{D} \alpha_{kij} \mathcal{N}(k|U^T V, \sigma^2) g'(\mu_i)(-1)^{[r_{ij}=0]} - \lambda_\mu \mu_i, \quad (18)$$

where $g'(x)$ is the derivative of the sigmoid function $g(x)$. In Eq. (16), (17), (18), $\lambda_u = \sigma^2/\sigma_u^2$, $\lambda_v = \sigma^2/\sigma_v^2$ and $\lambda_\mu = \sigma^2/\sigma_\mu^2$ and a multiplicative constant $1/\sigma^2$ is dropped in all three equations.

To learn model parameters, we alternatively update $U, V$ and $\mu$ using the gradient algorithm with a learning rate $\eta$ by maximizing the log-likelihood. First we update $U, V$ by

$$U_i \leftarrow U_i + \eta \frac{\partial \mathcal{L}}{\partial U_i}, \quad V_j \leftarrow V_j + \eta \frac{\partial \mathcal{L}}{\partial V_j}. \quad (19)$$

Then using the updated $U, V$, we update $\mu_i$ by

$$\mu_i \leftarrow \mu_i + \eta \frac{\partial \mathcal{L}}{\partial \mu_i}. \quad (20)$$

Similar to PMF [22], we linearly map the rating values in $[1, D]$ to $[0, 1]$ and pass $U_i^T V_j$ through the sigmoid function as defined in Eq. (13). To avoid cluttered notations, we drop all the logistic function in our derivation process. After obtained the trained model, we convert the expected value, $g(U_i^T V_j)$, back to the scale of 1 to $D$ and set it as the predicted score of user $i$’s rating on item $j$.

### 3.5 Context aware response model

In real-world recommender systems, the probability of an item being rated may not only depend on users’ rating score. Many factors affect the response probabilities. For example, in a movie rating system, some popular movies such as Titanic, Avatar, may have much higher probability of being rated than a mediocre movie. Moreover, the features of users and items may contain group structure [30]. One may argue this might be caused by the higher inspection rate, i.e., it is likely that a reputable movie is being watched more than an obscure one. Nevertheless, it still makes sense that some items may have higher chance of receiving a rating due to the high quality that a user will not hesitate to rate it. In addition, different user may have distinct rating habits. Some users might be more willing to provide ratings in order to get high quality