In both cases it will be helpful to know the number of fractions that are strictly less than \( \frac{k-1}{N+1} \) in \( \mathcal{P}_N \); this is

\[
\sum_{n=1}^{N} \sum_{m=1}^{\frac{k-1}{N+1}} = \sum_{n=1}^{\frac{(k-1)N}{N+1}} = \sum_{n=0}^{\frac{(k-1)n+N}{N+1}} = \frac{(k-2)N}{2} + \frac{d-1}{2} + d \left\lfloor \frac{N}{d} \right\rfloor = \frac{1}{2}(kn-d+1), \quad d = \gcd(k-1, N+1),
\]

by (3.32). Furthermore, the number of fractions equal to \( \frac{k-1}{N+1} \) in \( \mathcal{P}_N \) that should precede it in \( \mathcal{P}_{N+1} \) is \( \frac{1}{2}(d - 1 - [d \text{ even}]) \), by the nature of organ-pipe order.

If \( kn \) is odd, then \( d \) is even and \( \frac{k-1}{N+1} \) is preceded by \( \frac{1}{2}(kn-1) \) elements of \( \mathcal{P}_N \); this is just the correct number to make things work. If \( kn \) is even, then \( d \) is odd and \( \frac{k-1}{N+1} \) is preceded by \( \frac{1}{2}(kn) \) elements of \( \mathcal{P}_N \).

If \( d = 1 \), none of these equals \( \frac{k-1}{N+1} \) and \( \mathcal{P}_{N,kn} \) is '<'; otherwise \( \frac{k-1}{N+1} \) falls between two equal elements and \( \mathcal{P}_{N,kn} \) is '='. (C. S. Peirce [230] independently discovered the Stern-Brocot tree at about the same time as he discovered \( \mathcal{P}_N \).)

4.65 The analogous question for the (analogous) Fermat numbers \( f_n \) is a famous unsolved problem. This one might be easier or harder.

4.66 It is known that no square less than \( 36 \times 10^{18} \) divides a Mersenne number or Fermat number. But there has still been no proof of Schinzel's conjecture that there exist infinitely many squarefree Mersenne numbers. It is not even known if there are infinitely many \( p \) such that \( p\nmid(a \pm b) \), where all prime factors of \( a \) and \( b \) are \( \leq 31 \).

4.67 M. Szegedy has proved this conjecture for all large \( n \); see [284]', [77, pp. 78-79], and [49].

4.68 This is a much weaker conjecture than the result in the following exercise.

4.69 Cramér [56] showed that this conjecture is plausible on probabilistic grounds, and computational experience bears this out: Brent [32] has shown that \( P_{n+1} - P_n \leq 602 \) for \( P_{n+1} < 2.686 \times 10^{12} \). But the much weaker bounds in exercise 60 are the best currently proved [221]. Exercise 68 has a "yes" answer if \( P_{n+1} - P_n < 2P_n^{1/2} \) for all sufficiently large \( n \). According to Guy [139, problem A8], Paul Erdős offers $10,000 for proof that there are infinitely many \( n \) such that

\[
P_{n+1} - P_n > \frac{\ln n \ln \ln n \ln \ln \ln \ln n}{(\ln \ln n)^2}
\]

"No square less than \( 25 \times 10^{14} \) divides a Euclid number."

—Ilan Vardi