where \( k \) is an arbitrary nonnegative constant of integration. Thus the trajectories of the linear system (8) are ellipses centered at the critical point and elongated somewhat in the horizontal direction.

Now let us return to the nonlinear system (2). Dividing the second of Eqs. (2) by the first, we obtain

\[
\frac{dy}{dx} = y(-0.75 + 0.25x) / x(1 - 0.5y).
\]

Equation (12) is a separable equation and can be put in the form

\[
\frac{1 - 0.5y}{y} \frac{dy}{x} = -0.75 + 0.25x dx,
\]

from which it follows that

\[
0.75 \ln x + \ln y - 0.5y - 0.25x = c,
\]

where \( c \) is a constant of integration. Although by using only elementary functions we cannot solve Eq. (13) explicitly for either variable in terms of the other, it is possible to show that the graph of the equation for a fixed value of \( c \) is a closed curve surrounding the critical point \((3, 2)\). Thus the critical point is also a center of the nonlinear system (2) and the predator and prey populations exhibit a cyclic variation.

Figure 9.5.2 shows a phase portrait of the system (2). For some initial conditions the trajectory represents small variations in \( x \) and \( y \) about the critical point, and is almost elliptical in shape, as the linear analysis suggests. For other initial conditions the oscillations in \( x \) and \( y \) are more pronounced, and the shape of the trajectory is significantly different from an ellipse. Observe that the trajectories are traversed in the counterclockwise direction. The dependence of \( x \) and \( y \) on \( t \) for a typical set of initial conditions is shown in Figure 9.5.3. Note that \( x \) and \( y \) are periodic functions of \( t \), as they must be since the trajectories are closed curves. Further, the oscillation of the predator population lags behind that of the prey. Starting from a state in which both predator and prey populations are relatively small, the prey first increase because there is little predation. Then the predators, with abundant food, increase in population also. This causes heavier predation and the prey tend to decrease. Finally, with a diminished food supply, the predator population also decreases, and the system returns to the original state.

**FIGURE 9.5.2** A phase portrait of the system (2).