or
\[ \gamma^2 au \, du + \alpha^2 cv \, dv = 0. \]

Consequently,
\[ \gamma^2 au^2 + \alpha^2 cv^2 = k, \]
where \( k \) is a nonnegative constant of integration. Thus the trajectories of the linear system (17) are ellipses, just as in the example.

Returning briefly to the nonlinear system (1), observe that it can be reduced to the single equation
\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(-c + \gamma x)}{x(a - \alpha y)}. \]

Equation (21) is separable and has the solution
\[ a \ln y - \alpha y + c \ln x - \gamma x = C, \]
where \( C \) is a constant of integration. Again it is possible to show that the graph of Eq. (22), for fixed \( C \), is a closed curve surrounding the critical point \((c/\gamma, a/\alpha)\). Thus this critical point is also a center for the general nonlinear system (1).

The cyclic variation of the predator and prey populations can be analyzed in more detail when the deviations from the point \((c/\gamma, a/\alpha)\) are small and the linear system (17) can be used. The solution of the system (17) can be written in the form
\[ u = \frac{c}{\gamma} K \cos(\sqrt{ac} \, t + \phi), \quad v = \frac{a}{\alpha} \sqrt{\frac{c}{a}} K \sin(\sqrt{ac} \, t + \phi), \]
where the constants \( K \) and \( \phi \) are determined by the initial conditions. Thus
\[ x = \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{ac} \, t + \phi), \]
\[ y = \frac{a}{\alpha} + \frac{a}{\alpha} \sqrt{\frac{c}{a}} K \sin(\sqrt{ac} \, t + \phi). \]

These equations are good approximations for the nearly elliptical trajectories close to the critical point \((c/\gamma, a/\alpha)\). We can use them to draw several conclusions about the cyclic variation of the predator and prey on such trajectories.

1. The sizes of the predator and prey populations vary sinusoidally with period \(2\pi/\sqrt{ac}\). This period of oscillation is independent of the initial conditions.
2. The predator and prey populations are out of phase by one-quarter of a cycle. The prey leads and the predator lags, as explained in the example.
3. The amplitudes of the oscillations are \( Kc/\gamma \) for the prey and \( a\sqrt{c} K/\alpha \sqrt{a} \) for the predator and hence depend on the initial conditions as well as on the parameters of the problem.
4. The average populations of predator and prey over one complete cycle are \(c/\gamma\) and \(a/\alpha\), respectively. These are the same as the equilibrium populations; see Problem 10.

Cyclic variations of predator and prey as predicted by Eqs. (1) have been observed in nature. One striking example is described by Odum (pp. 191–192); based on the records of the Hudson Bay Company of Canada, the abundance of lynx and snowshoe hare as indicated by the number of pelts turned in over the period 1845–1935 shows...