a distinct periodic variation with period of 9 to 10 years. The peaks of abundance are followed by very rapid declines, and the peaks of abundance of the lynx and hare are out of phase, with that of the hare preceding that of the lynx by a year or more.

The Lotka–Volterra model of the predator–prey problem has revealed a cyclic variation that perhaps could have been anticipated. On the other hand, the use of the Lotka–Volterra model in other situations can lead to conclusions that are not intuitively obvious. An example that suggests a possible danger in using insecticides is given in Problem 12.

One criticism of the Lotka–Volterra equations is that in the absence of the predator the prey will grow without bound. This can be corrected by allowing for the natural inhibiting effect that an increasing population has on the growth rate of the population; for example, the first of Eqs. (1) can be modified so that when \( x \) is a logistic equation for \( x \) with population densities \( x \) and \( y \).

### PROBLEMS

Each of Problems 1 through 5 can be interpreted as describing the interaction of two species with population densities \( x \) and \( y \). In each of these problems carry out the following steps.

(a) Draw a direction field and describe how solutions seem to behave.

(b) Find the critical points.

(c) For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.

(d) Sketch the trajectories in the neighborhood of each critical point.

(e) Draw a phase portrait for the system.

(f) Determine the limiting behavior of \( x \) and \( y \) as \( t \to \infty \) and interpret the results in terms of the populations of the two species.

1. \[
   \begin{align*}
   \frac{dx}{dt} &= x(1.5 - 0.5y) \\
   \frac{dy}{dt} &= y(-0.5 + x)
   \end{align*}
   \]

2. \[
   \begin{align*}
   \frac{dx}{dt} &= x(1 - 0.5y) \\
   \frac{dy}{dt} &= y(-0.25 + 0.5x)
   \end{align*}
   \]

3. \[
   \begin{align*}
   \frac{dx}{dt} &= x(1 - 0.5x - 0.5y) \\
   \frac{dy}{dt} &= y(-0.25 + 0.5x)
   \end{align*}
   \]

4. \[
   \begin{align*}
   \frac{dx}{dt} &= x(1.125 - x - 0.5y) \\
   \frac{dy}{dt} &= y(-1 + x)
   \end{align*}
   \]

5. \[
   \begin{align*}
   \frac{dx}{dt} &= x(-1 + 2.5x - 0.3y - x^2) \\
   \frac{dy}{dt} &= y(-1.5 + x)
   \end{align*}
   \]

6. In this problem we examine the phase difference between the cyclic variations of the predator and prey populations as given by Eqs. (24) of the text. Suppose we assume that \( K > 0 \) and that \( t \) is measured from the time that the prey population \( x \) is a maximum; then \( \phi = 0 \). Show that the predator population \( y \) is a maximum at \( t = \pi/2\sqrt{ac} = T/4 \), where \( T \) is the period of the oscillation. When is the prey population increasing most rapidly, decreasing most rapidly, and when is it at some intermediate value? Answer the same questions for the predator population. Draw a typical elliptic trajectory enclosing the point \( (c/\gamma, a/\alpha) \), and mark these points on it.

7. (a) Find the ratio of the amplitudes of the oscillations of the prey and predator populations about the critical point \( (c/\gamma, a/\alpha) \), using the approximation (24), which is valid for small oscillations. Observe that the ratio is independent of the initial conditions.

(b) Evaluate the ratio found in part (a) for the system (2).

(c) Estimate the ratio of the amplitudes for the solution of the nonlinear system (2) shown in Figure 9.5.3. Does the result agree with that obtained from the linear approximation?