5.36 The sum of the digits of \( m + n \) is the sum of the digits of \( m \) plus the sum of the digits of \( n \), minus \( p - 1 \) times the number of carries, because each carry decreases the digit sum by \( p - 1 \).

5.37 Dividing the first identity by \( n! \) yields \( \binom{m+n}{n} = \sum_k \binom{m}{k} \binom{n}{n-k} \), Vandermonde’s convolution. The second identity follows, for example, from the formula \( x^k = (-1)^k |x|^k \) if we negate both \( x \) and \( y \).

5.38 Choose \( c \) as large as possible such that \( \binom{1}{0} \leq c \). Then \( 0 \leq c - \binom{1}{0} < \binom{c}{1} = \binom{c}{c-1} \); replace \( n \) by \( n - \binom{1}{0} \) and continue in the same fashion. Conversely, any such representation is obtained in this way. (We can do the same thing with
\[
\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}, \quad 0 \leq a_1 < a_2 < \cdots < a_m
\]
for any fixed \( m \).)

5.39 \( x^m y^n = \sum_{k=1}^n \binom{m+n-1-k}{n-1} a^n b^{m-k} x^k + \sum_{k=1}^n \binom{m+n-1-k}{m-1} a^n b^{m-k} y^k \) for all \( m, n > 0 \), by induction on \( m + n \).

5.40 \( (-1)^{-m+1} \sum_{k=1}^n \sum_{j=1}^m \binom{m-k-1}{n-j} \binom{m-1}{k} \binom{n}{j} = (-1)^{m+1} \sum_{k=1}^n \binom{m-1}{n-k} \binom{m+1}{m-k} \binom{n+k-1}{m} - \binom{m-1}{m} \binom{n+k-1}{m} \), which is \( 2^m n!/(2n+1)! \).

5.41 \( \sum_{k=0}^n n!/(n-k)! (n + k + 1)! \equiv (n!/(2n+1)! \sum_{k=n}^{2n+1} \binom{2n+1}{k} \), which is \( 2^{2n} n!/(2n+1)! \).

5.42 We treat \( n \) as an indeterminate real variable. Gosper’s method with \( q(k) = k + 1 \) and \( r(k) = k + 1 - n \) has the solution \( s(k) = 1/(n + 2) \); hence the desired indefinite sum is \( (-1)^x \frac{x+1}{n+2} / \binom{x+1}{n+1} \). And
\[
\frac{\sum_{k=0}^n (-1)^k / \binom{n}{k}}{\binom{n+1}{n+2}} = \frac{1}{\binom{n+1}{n+2}} \frac{1}{\binom{n+1}{x+1}} = \frac{2}{n+2} [n \text{ even}].
\]

This exercise, incidentally, implies the formula
\[
\frac{1}{n \binom{n-1}{k}} = \frac{1}{(n+1) \binom{n}{k+1}} + \frac{1}{(n+1) \binom{n}{k}},
\]
a “dual” to the basic recurrence (5.8).

5.43 After the hinted first step we can apply (5.21) and sum on \( k \). Then (5.21) applies again and Vandermonde’s convolution finishes the job. (A combinatorial proof of this identity has been given by Andrews [10]. There’s a quick way to go from this identity to a proof of (5.29), explained in [173, exercise 1.2.6-62].)