and this is the case \( x = N \) of exercise 22.

5.55 Let \( Q(k) = (k + A_1) \cdots (k + A_M)Z \) and \( R(k) = (k + B_1) \cdots (k + B_N) \). Then \( t(k + 1)/t(k) = P(k)|Q(k - 1)/P(k - 1)R(k) \), where \( P(k) = Q(k) - R(k) \) is a nonzero polynomial.

5.56 The solution to \( -(k + 1)(k + 2) = s(k + 1) + s(k) \) is \( s(k) = -\frac{1}{2}k^2 - \frac{1}{4}k \); hence \( \sum \binom{n}{k} \delta k = \frac{1}{8}(-1)^{k-1}(2k^2 + 4k + 1) + C \). Also

\[
(-1)^{k-1}\begin{bmatrix} \frac{k + 1}{2} \\ \frac{k + 2}{2} \end{bmatrix} = \frac{(-1)^{k-1}}{4} \left( k + 1 - \frac{1 + \frac{(-1)^k}{2}}{2} \right) \left( k + 2 - \frac{1 - \frac{(-1)^k}{2}}{2} \right)
= \frac{(-1)^{k-1}}{8}(2k^2 + 4k + 1) + \frac{1}{8}
\]

5.57 We have \( t(k + 1)/t(k) = (k - n)(k + 1 + \theta)(-z)/(k + l)(k + \theta) \). Therefore we let \( p(k) = k + \theta, q(k) = (k - n)(-z), r(k) = k \). The secret function \( s(k) \) must be a constant \( \alpha_0 \), and we have

\[ k + \theta = (-z(k - n) - k) \alpha_0 \]

hence \( \alpha_0 = -1/(1 + z) \) and \( \theta = -nz/(1 + z) \). The sum is

\[
\sum \binom{n}{k} z^k \frac{-(n - 1)}{1 + z} \delta k = -\frac{n}{1 + z} \binom{n - 1}{k - 1} z^k + C
\]

(The special case \( z = 1 \) was mentioned in (5.18); the general case is equivalent to (5.131).)

5.58 If \( m > 0 \) we can replace \( \binom{k}{m} \) by \( \frac{k}{m} \binom{k - 1}{m - 1} \) and derive the formula \( T_{m,n} = \frac{m!}{n!} T_{m-1,n-1} \frac{1}{m} \binom{n}{m} \). The summation factor \( \binom{n}{m}^{-1} \) is therefore appropriate:

\[
T_{m,n} / \binom{n}{m} = T_{m-1,n-1} \frac{1}{m} + \frac{1}{n}
\]

We can unfold this to get

\[
T_{m,n} / \binom{n}{m} = T_{0,n-m} H_m + T_{n,n-m} H_n - H_{n-m}.
\]

Finally \( T_{0,n-m} = H_{n-m} \), so \( T_{m,n} = \binom{n}{m} (H_n - H_m) \). (It’s also possible to derive this result by using generating functions; see Example 2 in Section 7.5.)

5.59 \( \sum_{j \geq 0, k \geq 1} \binom{n}{j} [j = [\log_m k]] = \sum_{j \geq 0, k \geq 1} \binom{n}{j} [m^j \leq k < m^{j+1}] \), which is \( \sum_{j \geq 0} \binom{n}{j} (m^j - m^{j-1}) = (m - 1) \sum_{j \geq 0} \binom{n}{j} m^j = (m - 1)(m + 1)^n \).