We also want to consider the function
\[ V(x, y) = V_x(x, y)F(x, y) + V_y(x, y)G(x, y), \]
where \( F \) and \( G \) are the same functions as in Eqs. (6). We choose this notation because \( V(x, y) \) can be identified as the rate of change of \( V \) along the trajectory of the system (6) that passes through the point \((x, y)\). That is, if \( x = \phi(t), y = \psi(t) \) is a solution of the system (6), then
\[
\frac{dV[\phi(t), \psi(t)]}{dt} = V_x[\phi(t), \psi(t)] \frac{d\phi(t)}{dt} + V_y[\phi(t), \psi(t)] \frac{d\psi(t)}{dt} \\
= V_x(x, y)F(x, y) + V_y(x, y)G(x, y) \\
= \dot{V}(x, y).
\]

(8)

The function \( \dot{V} \) is sometimes referred to as the derivative of \( V \) with respect to the system (6).

We now state two Liapunov theorems, the first dealing with stability, the second with instability.

**Theorem 9.6.1**
Suppose that the autonomous system (6) has an isolated critical point at the origin. If there exists a function \( V \) that is continuous and has continuous first partial derivatives, is positive definite, and for which the function \( \dot{V} \) given by Eq. (7) is negative definite on some domain \( D \) in the \( xy \)-plane containing \((0, 0)\), then the origin is an asymptotically stable critical point. If \( \dot{V} \) is negative semidefinite, then the origin is a stable critical point.

**Theorem 9.6.2**
Let the origin be an isolated critical point of the autonomous system (6). Let \( V \) be a function that is continuous and has continuous first partial derivatives. Suppose that \( V(0, 0) = 0 \) and that in every neighborhood of the origin there is at least one point at which \( V \) is positive (negative). If there exists a domain \( D \) containing the origin such that the function \( \dot{V} \) given by Eq. (7) is positive definite (negative definite) on \( D \), then the origin is an unstable critical point.

The function \( V \) is called a Liapunov function. Before sketching geometrical arguments for Theorems 9.6.1 and 9.6.2, we note that the difficulty in using these theorems is that they tell us nothing about how to construct a Liapunov function, assuming that one exists. In cases where the autonomous system (6) represents a physical problem, it is natural to consider first the actual total energy function of the system as a possible Liapunov function. However, Theorems 9.6.1 and 9.6.2 are applicable in cases where the concept of physical energy is not pertinent. In such cases a judicious trial-and-error approach may be necessary.

Now consider the second part of Theorem 9.6.1, that is, the case \( \dot{V} \leq 0 \). Let \( c \geq 0 \) be a constant and consider the curve in the \( xy \)-plane given by \( V(x, y) = c \). For \( c = 0 \) the curve reduces to the single point \( x = 0, y = 0 \). We assume that if \( 0 < c_1 < c_2 \), then the curve \( V(x, y) = c_1 \) contains the origin and lies within the curve \( V(x, y) = c_2 \), as illustrated in Figure 9.6.1a. We show that a trajectory starting inside a closed curve \( V(x, y) = c \) cannot cross to the outside. Thus, given a circle of radius \( \epsilon \) about the