9.6 Liapunov’s Second Method

Let \( V(x, y) \) be a point of intersection of the trajectory and a closed curve \( V \) at worst, tangent to this curve. Trajectories starting inside a closed curve \( V \) are tangent to the trajectory at each point; see Figure 9.6.1b. Thus the direction of motion on the trajectory is inward with respect to the closed curve \( V \) and points away from the origin, as indicated in Figure 9.6.1b. Next, consider a trajectory \( x = \phi(t), \ y = \psi(t) \) of the system (6) and recall that the vector \( T(t) = \phi'(t)i + \psi'(t)j \) is tangent to the trajectory at each point; see Figure 9.6.1b. Let \( x_1 = \phi(t_1), \ y_1 = \psi(t_1) \) be a point of intersection of the trajectory and a closed curve \( V(x, y) = c \). At this point \( \phi'(t_1) = F(x_1, y_1), \psi'(t_1) = G(x_1, y_1) \), so from Eq. (7) we obtain

\[
\dot{V}(x_1, y_1) = V_x(x_1, y_1)\phi'(t_1) + V_y(x_1, y_1)\psi'(t_1)
= [V_x(x_1, y_1)i + V_y(x_1, y_1)j] \cdot [\phi'(t_1)i + \psi'(t_1)j]
= \nabla V(x_1, y_1) \cdot T(t_1).
\] (10)

Thus \( \dot{V}(x_1, y_1) \) is the scalar product of the vector \( \nabla V(x_1, y_1) \) and the vector \( T(t_1) \). Since \( \dot{V}(x_1, y_1) \leq 0 \), it follows that the cosine of the angle between \( \nabla V(x_1, y_1) \) and \( T(t_1) \) is also less than or equal to zero; hence the angle itself is in the range \([\pi/2, 3\pi/2]\). Thus the direction of motion on the trajectory is inward with respect to \( V(x_1, y_1) = c \) or, at worst, tangent to this curve. Trajectories starting inside a closed curve \( V(x_1, y_1) = c \) (no matter how small \( c \) is) cannot escape, so the origin is a stable point. If \( \dot{V}(x_1, y_1) < 0 \), then the trajectories passing through points on the curve are actually pointed inward. As a consequence, it can be shown that trajectories starting sufficiently close to the origin must approach the origin; hence the origin is asymptotically stable.

A geometric argument for [Theorem 9.6.2] follows in a somewhat similar manner. Briefly, suppose that \( V \) is positive definite, and suppose that given any circle about the origin there is an interior point \((x_1, y_1)\) at which \( V(x_1, y_1) > 0 \). Consider a trajectory that starts at \((x_1, y_1)\). Along this trajectory it follows from Eq. (8) that \( V \) must increase, since \( \dot{V}(x_1, y_1) > 0 \); furthermore, since \( V(x_1, y_1) > 0 \), the trajectory cannot approach the origin because \( V(0, 0) = 0 \). This shows that the origin cannot be asymptotically stable.