stable. By further exploiting the fact that \( \dot{V}(x, y) > 0 \), it is possible to show that the origin is an unstable point; however, we will not pursue this argument.

**Example 2**

Use Theorem 9.6.1 to show that \((0, 0)\) is a stable critical point for the undamped pendulum equations (2). Also use Theorem 9.6.2 to show that \((\pi, 0)\) is an unstable critical point.

Let \( V \) be the total energy given by Eq. (4):

\[
V(x, y) = mgL(1 - \cos x) + \frac{1}{2}mL^2y^2.
\]  

(4)

If we take \( D \) to be the domain \(-\pi/2 < x < \pi/2, -\infty < y < \infty\), then \( V \) is positive there except at the origin, where it is zero. Thus \( V \) is positive definite on \( D \). Further, as we have already seen,

\[
\dot{V} = (mgL \sin x)(y) + (mL^2y)(-g \sin x)/L = 0
\]

for all \( x \) and \( y \). Thus \( \dot{V} \) is negative semidefinite on \( D \). Consequently, by the last statement in Theorem 9.6.1, the origin is a stable critical point for the undamped pendulum. Observe that this conclusion cannot be obtained from Theorem 9.3.2 because \((0, 0)\) is a center for the corresponding linear system.

Now consider the critical point \((\pi, 0)\). The Liapunov function given by Eq. (4) is no longer suitable because Theorem 9.6.2 calls for a function \( V \) for which \( \dot{V} \) is either positive or negative definite. To analyze the point \((\pi, 0)\) it is convenient to move this point to the origin by the change of variables \( x = \pi + u, y = v \). Then the differential equations (2) become

\[
\frac{du}{dt} = v, \quad \frac{dv}{dt} = \frac{g}{L} \sin u,
\]  

(11)

and the critical point is \((0, 0)\) in the \( uv \)-plane. Consider the function

\[
V(u, v) = v \sin u
\]  

(12)

and let \( D \) be the domain \(-\pi/4 < u < \pi/4, -\infty < v < \infty\). Then

\[
\dot{V} = (v \cos u)(v) + (\sin u)[(g/L) \sin u] = v^2 \cos u + (g/L) \sin^2 u
\]  

(13)

is positive definite in \( D \). The only remaining question is whether there are points in every neighborhood of the origin where \( V \) itself is positive. From Eq. (12) we see that \( V(u, v) > 0 \) in the first quadrant (where both \( \sin u \) and \( v \) are positive) and in the third quadrant (where both are negative). Thus the conditions of Theorem 9.6.2 are satisfied and the point \((0, 0)\) in the \( uv \)-plane, or the point \((\pi, 0)\) in the \( xy \)-plane, is unstable.

The damped pendulum equations are discussed in Problem 7.

From a practical point of view we are often more interested in the basin of attraction. The following theorem provides some information on this subject.

**Theorem 9.6.3**

Let the origin be an isolated critical point of the autonomous system (6). Let the function \( V \) be continuous and have continuous first partial derivatives. If there is a bounded domain \( D_K \) containing the origin where \( V(x, y) < K \), \( V \) is positive definite,